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An impulsive input approach to short time convergent control for linear systems†

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Abstract The paper considers the problem of bringing the state of a controllable linear system to the origin in a very short time. It takes the approach of considering an "ideal" control input consisting of a linear combination of the Dirac delta function and its derivatives that realizes this goal instantaneously. Three schemes are introduced to approximate the impulsive input with physically realizable functions: a smooth approximation with compact support, a Gaussian function approximation and a step approximation. It is shown using a numerical example that all approximations work reasonably well, with the Gaussian approximation providing slightly worse results. It is also shown that a direct approach to obtain a state nulling input by solving an integral equation runs quicker into numerical problems than the impulsive input approach as the convergence time decreases. Finally, an application to an orbital rendez-vous problem is presented.

1 Introduction and motivation

The interest in impulsive control theory has steadily increased over the past few years with many new books and articles being added to an already impressive list. Without any ambition to be exhaustive we may cite here books like [1, 2, 3, 4] and numerous journal and conference contributions such as [5, 6, 7]. The idea of using the delta distribution and its derivatives in control synthesis is not new. This approach seems to be first considered in [8]. Another work that takes a similar approach is [9]. More recent publications such as [7] have even extended the problem to the case of linear descriptor systems. In the paper [5] a dynamic programming

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approach is proposed for optimal impulsive control laws that turn out to be linear combinations of delta and delta derivatives. Another recent contribution [6] uses delta and delta derivative state feedback for the adaptive stabilization of a second order nonlinear system.

There are numerous practical situations in which impulsive control is not just an option, but the only solution to achieve the required performance. In general, this is the case in all situations that large deviations from equilibrium need to be corrected in very short time. An example in this direction are the reaction control systems for steering and attitude control of space vehicles. For instance, see the works [10], [6] and [11]. For exo-atmospheric missiles, reaction control based on solid fuel rocket thrusters is an attractive solution, but they are not throttable and deliver a large impulse during a short time period. The action of small thrusters on the missile can largely be approximated by an impulsive signal.

Let us illustrate the approach of this paper using a simple example of a perturbed double-integrator

$$\begin{aligned} \dot{x}_1 &= x_{2,} \\ \dot{x}_2 &= f(t) + u, \end{aligned} \tag{1}$$

with a control input u and an unknown disturbance $f(t)$. This may be the model of a high precision positioning system of a point mass, with x_1, the position, and x_2, the velocity. Typically a fine positioning control system will use high accuracy sensors and actuators to ensure that the disturbing force is effectively rejected. However, high accuracy sensors and actuators typically have a limited range. The system is designed in such a way that, most of the time, the disturbance will not bring the system out of the range of the fine positioning control system, except if a peak in the disturbing force occurs. These occurrences may be rare, but a fine positioning control system will be poorly equipped to deal with these situations, so an additional system needs to be in place.

In this situation it is important to restore as quickly as possible the system to the neighborhood of the origin, so that the fine control system can take it over. This is in essence the problem that we consider in this paper.

As the occurrence of peaks in the disturbance is assumed to be a rare event, we are not very much concerned with limitations in the energy necessary to perform the correction. We will even allow for impulsive inputs. For example, an impulsive input of the form

$$u = x_2(0)\delta_0 + x_1(0)\dot{\delta}_0$$

brings the double integrator (1) instantaneously to the origin. Of course, this input needs to be practically implemented, and we are examining this problem in this paper too. We limit ourselves here to the open loop control problem, leaving the feedback control for future work.

One of the main problems in impulsive control consists is in the approximation of the delta function and its derivatives. The most popular approximation is one based on the Gaussian bell function as used in [9] and [7]. In this paper, we compare this approximation with two other approximations that have the advantage of having finite support. The main goal and the contribution of this paper is to present a theoretical and practical study of these approximations of the delta function and its derivatives and to compare these approximations with a direct approach to bring the state of the system to the origin that is based on a solution of an integral Volterra equation. We will see that the Gaussian function approximation provides slightly worse results in practice, whereas the direct approach runs into serious numerical problems as the convergence time decreases.

The structure of the paper is as follows. Next section formulates the problem that we consider in this paper: nulling the state of the system in very short time using an impulsive type of input. Section 3 presents the main theoretical contribution of the paper, deriving the formula's for the ideal impulsive input, as well as offering three different solutions to approximate the impulsive inputs with practically implementable signals. These solutions are compared to a conventional solution both qualitatively, as on a numerical example in Section 4. In Section 5, the proposed techniques are illustrated on a satellite rendezvous problem. Finally, Section 6 presents some concluding remarks and ideas for future work.

2 Problem formulation

Consider the following linear system with n states and m inputs:

$$\dot{x} = Ax + Bu,\; x(0) = x_0. \tag{2}$$

The problem that we consider in this paper can broadly be formulated as: *Find an input signal u that brings the state to the origin in a short time*. Of course, the problem formulated in this way has very many solutions. In fact, it has many ways to approach it.

One of the well-known approaches is the Minimum Time Optimal Control Problem, often used as an application of the Optimum Principle of Pontriaghin. In this case, it is assumed that there is a bound on the magnitude of the control signal u and the problem is to find an admissible input u that brings the state in the origin in minimum time.

This is not however the approach that we take here. In fact, we will not put any bound on the magnitude of u, but we will rather fix a time interval within which the state should be nulled. By making this time interval arbitrarily short, we hope to achieve the stated objective. Even under this formulation of the problem, there are infinitely many solutions. The most straightforward approach, that we will call the *direct approach*, consists of determining an input that solves the integral equation of Volterra-type

$$0 = \exp(A\varepsilon)x_0 + \int_0^{\varepsilon} \exp(A(\varepsilon - s))Bu(s)ds, \tag{3}$$

that can be deduced directly by imposing $x(\varepsilon) = 0$ and using the variations of constants formula for (2). If the system (2) is controllable (an obviously necessary condition for the problem to have a solution at all), there are infinitely many solutions of this equation that can be generated in the following way: Let $Q(\cdot)$ be an $m \times m$-matrix valued function such that

$$W_\varepsilon = \int_0^\varepsilon \exp(A(\varepsilon - s))BQ(s)B^T \exp(A^T(\varepsilon - s))ds \tag{4}$$

is invertible. Then

$$u(t) = -Q(t)B^T \exp(A^T(\varepsilon - t))W_\varepsilon^{-1} \exp(A\varepsilon)x_0 \tag{5}$$

is a solution of (3), as can be easily verified. In this way, we constructed an entire set of solutions for the equation (3). Each of them represent an input function that brings the state of (2) in the origin at time ε.

The approach that we introduce in this paper and that we call the *impulsive input approach* is based on starting with an input u that brings the state in the origin instantaneously. Of course, such an input signal necessarily has an impulsive character. In fact, it is a sum of Dirac delta derivatives. By approximating the Dirac delta derivatives, we can determine practical input signals that “almost” bring the state in the origin. We will actually present two different systematic ways of determining an approximation for the impulsive input.

3 The impulsive input approach

As explained before, we are looking for an input of the form

$$u(t) = \sum_{k=0}^{n-1} \delta_\varepsilon^{(k)}(t)\alpha_k, \tag{6}$$

where $\delta_\varepsilon^{(k)}$ are the generalized derivatives of the Dirac-delta distribution centered in $\varepsilon > 0$, defined (see e.g. [12, Sec. 2.2]) as

$$\int \delta_\varepsilon^{(k)}(t)\phi(t)dt = (-1)^k \phi^{(k)}(\varepsilon),$$

for any test function ϕ, and α_k are vectors of dimension m, that need to be determined. By substituting (6) in the variation-of-constants formula for system (2), we have

$$x(t) = \exp(At)x_0 + \sum_{k=0}^{n-1} [\int_0^t \exp(A(t-s))B\delta_\varepsilon^{(k)}(s)ds]\alpha_k.$$

Using familiar properties of derivatives of the Dirac-delta distribution, this is equivalent to

$$x(t) = \exp(At)x_0 + \sum_{k=0}^{n-1} \exp(A(t-\varepsilon))A^k B\alpha_k,$$

for $t \geq \varepsilon$. Requiring that $x(\varepsilon) = 0$, the coefficients α_k need to satisfy

$$\begin{bmatrix} B\ AB\ \ldots\ A^{n-1}B \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{bmatrix} = -\exp(A\varepsilon)x_0. \tag{7}$$

Notice that the matrix in the left hand side of (7) is the controllability matrix of the pair (A,B). If system (2) is controllable, (7) has a least one solution for every x_0 and every ε. For each such solution, the input (6) will make $x(\varepsilon) = 0$.

Obviously, to make such an approach practical, it is necessary to approximate the impulsive input with a regular input. Fortunately, this is possible. In fact, there are infinitely many ways to do this. We will propose here two types of approximations: using smooth functions (one with bounded support and one using the Gaussian function) and using step functions.

3.1 A C^∞ approximation with bounded support

Consider the following kernel function

$$\omega_h(t) = \begin{cases} \frac{1}{\kappa h} e^{\frac{t^2}{t^2-h^2}}, & |t| < h, \\ 0 & |t| \geq h, \end{cases}$$

where $\kappa = \int_{-1}^{1} e^{\frac{t^2}{t^2-1}} dt$ is a normalization factor, and $h > 0$ is arbitrary. It is well known that the functions ω_h are C^∞ smooth, and as $h \to 0$, these functions approximate in a special sense the Dirac-delta distribution (i.e. they weakly converge to the delta distribution, see e.g. [13, pag. 13 and following]).

We propose to replace the input (6) by

$$u_h(t) = \sum_{k=0}^{n-1} \omega_h^{(k)}(t-\varepsilon)\alpha_k. \tag{8}$$

For any $\varepsilon > h > 0$, this function is smooth and is null everywhere outside the interval $[\varepsilon - h, \varepsilon + h]$. Due to the approximation property of the kernel function, it is to be expected that the state response will come close to the origin for $t \geq h + \varepsilon$.

Proposition 1 *Let $x_h(\cdot)$ denote the solution of (2) for $u = u_h$, for some positive h. Then*

$$\lim_{h\to 0} x_h(h+\varepsilon) = 0.$$

Proof. Introducing (8) into

$$x_h(h+\varepsilon) = \exp(A(h+\varepsilon))x_0 + \int_0^{h+\varepsilon} \exp(A(h+\varepsilon-s))Bu_h(s)ds,$$

and using the formula

$$\int_0^{h+\varepsilon} \exp(A(h+\varepsilon-\tau))B\alpha\omega_h^{(k)}(\tau-\varepsilon)d\tau = \int_0^{h+\varepsilon} \exp(A(h+\varepsilon-\tau))A^kB\alpha\omega_h(\tau-\varepsilon)d\tau,$$

that can be proven by using integration by parts and induction, we obtain

$$x_h(h+\varepsilon) = \exp(A(h+\varepsilon))x_0 + \int_0^{h+\varepsilon} \exp(A(h+\varepsilon-\tau))\sum_{k=0}^{n-1} A^kB\alpha_k\omega_h(\tau-\varepsilon)d\tau.$$

By substituting here formula (7) we obtain

$$x_h(h+\varepsilon) = \exp(A(h+\varepsilon))x_0 - \int_0^{h+\varepsilon} \exp(A(h+2\varepsilon-\tau))x_0\omega_h(\tau-\varepsilon)d\tau.$$

Because of the finite support of the function ω_h, the limits of integration in the previous expression can be extended to the entire axis. Now, using the fact that, for every continuous function $\phi(t)$

$$\lim_{h\to 0}\int_{-\infty}^{\infty} \phi(t-\tau)\omega_h(\tau)d\tau = \phi(t),$$

we can take directly the limit in the last expression of $x_h(h+\varepsilon)$ and the assertion is readily proved.

3.2 A Gaussian function approximation

The approximation of the impulsive input by the Gaussian function approximation was proposed and studied in quite a few references (e.g. [8, 7]). In this case, the Dirac delta function is approximated as

$$\Phi_h(t) = \frac{1}{\sqrt{2\pi}h}e^{\frac{t^2}{2h^2}}.$$

Although, this is a C^∞ function, unlike the previous approximation, this function does not have compact support. A similar result as Proposition 1 holds for this approximation, but we will not state here since this approximation was extensively studied in the literature.

3.3 A piecewise-constant function approximation

The function

$$\delta_h^{[0]}(t) = \begin{cases} \frac{1}{2h}, & |t| \leq h \\ 0, & rest. \end{cases} \tag{9}$$

is clearly a piecewise constant approximation of the Dirac delta function. The first order derivative of the delta function can be approximated by the following "symmetric finite difference" relation

$$\delta_h^{[1]} = \frac{\delta_{\frac{h}{2}}^{[0]}(t+\frac{h}{2}) - \delta_{\frac{h}{2}}^{[0]}(t-\frac{h}{2})}{h} = \begin{cases} \frac{1}{h^2}, & -h \leq t \leq 0, \\ -\frac{1}{h^2}, & 0 < t \leq h, \\ 0, & rest. \end{cases} \tag{10}$$

Notice that the support of $\delta_h^{[1]}$ as defined above, just as the support of $\delta_h^{[0]}$ is $[-h,h]$. Approximations of the higher order derivatives are defined iteratively as

$$\delta_h^{[k]}(t) = \frac{\delta_{\frac{h}{2}}^{[k-1]}(t+\frac{h}{2}) - \delta_{\frac{h}{2}}^{[k-1]}(t-\frac{h}{2})}{h}, \tag{11}$$

for all $k \geq 1$.

We propose to replace the input (6) by

$$u_h(t) = \sum_{k=0}^{n-1} \delta_h^{[k]}(t-\varepsilon)\alpha_k. \tag{12}$$

Just as the approximation (8), for any $\varepsilon \geq h > 0$, this function is null everywhere outside the interval $[\varepsilon - h, \varepsilon + h]$. The next result shows that this input is also bringing the state close to the origin for $t \geq h + \varepsilon$ for h small enough.

Proposition 2 *Let $x_h(\cdot)$ denote the solution of (2) for $u = u_h$, for some positive h. Then*

$$\lim_{h\to 0} x_h(h+\varepsilon) = 0.$$

Proof. Introducing (12) into

$$\begin{aligned} x_h(h+\varepsilon) = {} & \exp(A(h+\varepsilon))x_0 + \\ & + \int_0^{h+\varepsilon} \exp(A(h+\varepsilon-s))Bu_h(s)ds, \end{aligned}$$

while taking into account that the support of u_h is $[h+\varepsilon, \varepsilon-h]$,

$$x_h(h+\varepsilon) = \exp(A(h+\varepsilon))x_0 + \sum_{k=0}^{n-1} \alpha_k \int_{-h+\varepsilon}^{h+\varepsilon} e^{A(h+\varepsilon-s)} B\delta_h^{[k]}(s-\varepsilon)ds,$$

and after a simple change of variable

$$x_h(h+\varepsilon) = e^{A(h+\varepsilon)}x_0 + \sum_{k=0}^{n-1} \alpha_k \int_{-h}^{h} e^{A(h-s)} B\delta_h^{[k]}(s)ds.$$

From the last relation and from (7), it is clear that the assertion is proved if we show that

$$\lim_{h\to 0} \int_{-h}^{h} \exp(A(h-s))B\delta_h^{[k]}(s)ds = A^k B. \tag{13}$$

We prove this relation by induction. First of all, for $k=0$, the relation

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} \exp(A(h-s))Bds = B$$

follows from the properties of the matrix exponential.

Let us denote by

$$I_k(h) = \int_{-h}^{h} \exp(A(h-s))B\delta_h^{[k]}(s)ds. \tag{14}$$

By hypothesis

$$\lim_{h\to 0} I_{k-1}(h) = A^{k-1}B.$$

Using the recursive definition (11), we can write

$$I_k(h) = \frac{1}{h} \int_{-h}^{h} e^{A(h-s)} B[\delta_{\frac{h}{2}}^{[k-1]}(s+\frac{h}{2}) - \delta_{\frac{h}{2}}^{[k-1]}(s-\frac{h}{2})]ds.$$

Taking into account that the support of $\delta_{\frac{h}{2}}^{[k-1]}$ is $[-\frac{h}{2}, \frac{h}{2}]$, we have

$$I_k(h) = \frac{1}{h}[\int_{-h}^{0} \exp(A(h-s))B\delta_{\frac{h}{2}}^{[k-1]}(s+\frac{h}{2})ds - \int_{0}^{h} \exp(A(h-s))B\delta_{\frac{h}{2}}^{[k-1]}(s-\frac{h}{2})ds],$$

and translating the variable in each integral,

$$I_k(h) = \frac{1}{h}[\int_{-\frac{h}{2}}^{\frac{h}{2}} \exp(A(\frac{3h}{2}-s))B\delta_{\frac{h}{2}}^{[k-1]}(s)ds$$
$$- \int_{-\frac{h}{2}}^{\frac{h}{2}} \exp(A(\frac{h}{2}-s))B\delta_{\frac{h}{2}}^{[k-1]}(s)ds],$$

that is

$$I_k(h) = \frac{1}{h}[\exp(Ah)-I]\int_{-\frac{h}{2}}^{\frac{h}{2}} e^{A(\frac{h}{2}-s)}B\delta_{\frac{h}{2}}^{[k-1]}(s)ds$$
$$= \frac{1}{h}[\exp(Ah)-I]I_{k-1}(\frac{h}{2}).$$

Since

$$\lim_{h\to 0}\frac{1}{h}[\exp(Ah)-I] = A,$$

it is clear that

$$\lim_{h\to 0} I_k(h) = A^k B,$$

and the proof is completed.

4 comparing the impulsive input approach with the direct approach

The approach based on approximating the impulsive input requires very simple computations, and they are independent of the convergence time h. Formula (12) can be computed by solving a linear equation (7) and using analytic functions such as the function ω_h and its derivatives that can be even computed off-line. Similarly, the approximation using piecewise constant function and the approximation using Gaussian functions can easily be implemented using appropriate data computed off-line. The piecewise constant approximation may be easier to implement in a practical situation, and presents a definite advantage that it allows a good estimate of the maximum value of the control input. On the other hand, formula (5) requires numerical integration to compute W_h and then matrix exponentials are also required.

However, formula (12) is only an approximate solution of the state nulling problem, whereas formula (5) is an exact solution. Also, any choice of $Q(t)$ that keeps W_h invertible generates a solution to the state nulling problem. Notice that equation (7) may also have an infinity of solutions in case $m > 1$, that is, if there are more than a single input.

So far the qualitative analysis. We tested the various approximations of the impulsive control as well as the direct approach on a few numerical examples. Due to space limitation, we present only a single example of a third order system with one input:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & -2 & 1 \\ 3 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, x_0 = \begin{bmatrix} 30 \\ -20 \\ 42 \end{bmatrix}.$$

The results for the first, impulsive input approach are represented in Figures 1 and 2 for the case of the C^∞ approximation and in Figures 3 and 4 for the case of the piecewise constant approximation. The results for the Gaussian approximation are represented in Figures 5 and 6. Notice that the Gaussian approximation provided worse results than the other two approximations especially as the convergence time decreases. We have seen this for all the examples considered, and it is probably due to the fact that the Gaussian function has no compact support, but we do not have at this time a rigorous argument to explain this observation.

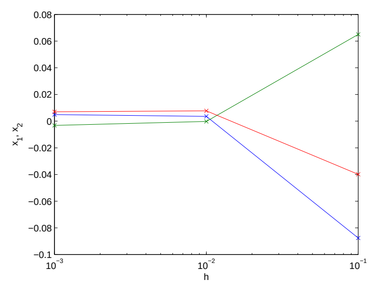


Fig. 1: State value after applying the impulsive input approach with C^∞approximation for different times h.

Applying the direct approach on the same example, we obtain the results represented in Figures 7 and 8. This is much worse than expected. Actually, the state is effectively brought close to the origin only for the case of $h = 0.1$, in which case the state coordinates are of the order 0.01. However, this is not visible in Figure 7 due to the very bad performance for the other two values of h.

Analyzing the cause of the failure of the direct approach in this case, we notice that formula (5) involves the inverse of the matrix W_h, which even if invertible for

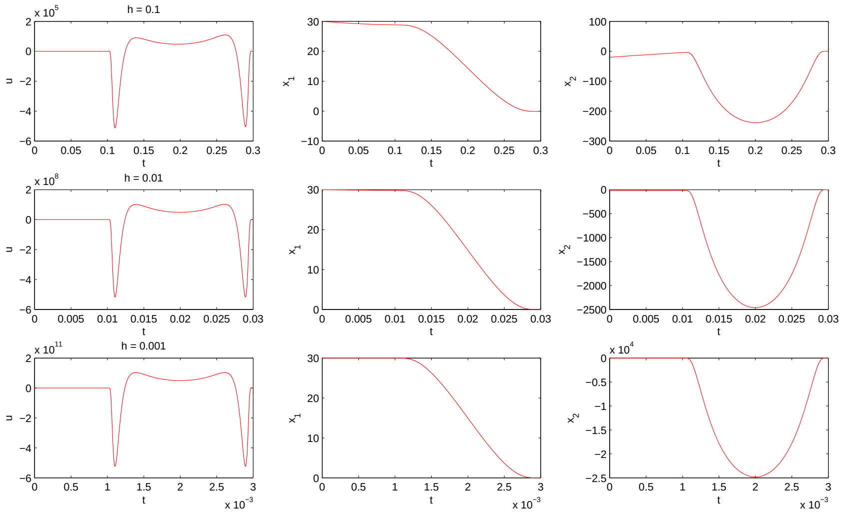


Fig. 2: Input and state response using the impulsive input approach with C^{∞}approximation for different times h.

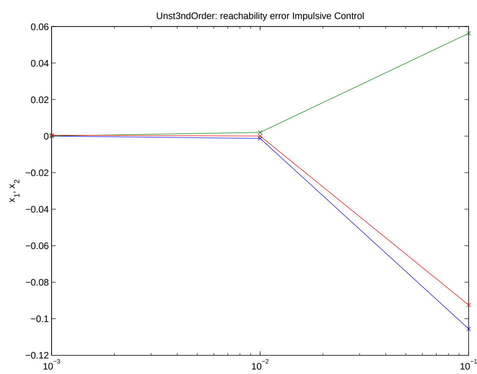


Fig. 3: State value after applying the impulsive input approach with piecewise constant approximation for different times h.

all $h > 0$ in case that the pair (A,B) is controllable, may actually be quite poorly conditioned. In this case, the computation of the nulling input will be challenging. Let us consider

$$\tilde{W}_h = \int_0^h \exp(-As)BB^T \exp(-A^T s))ds,$$

which is related to W_h by the relation

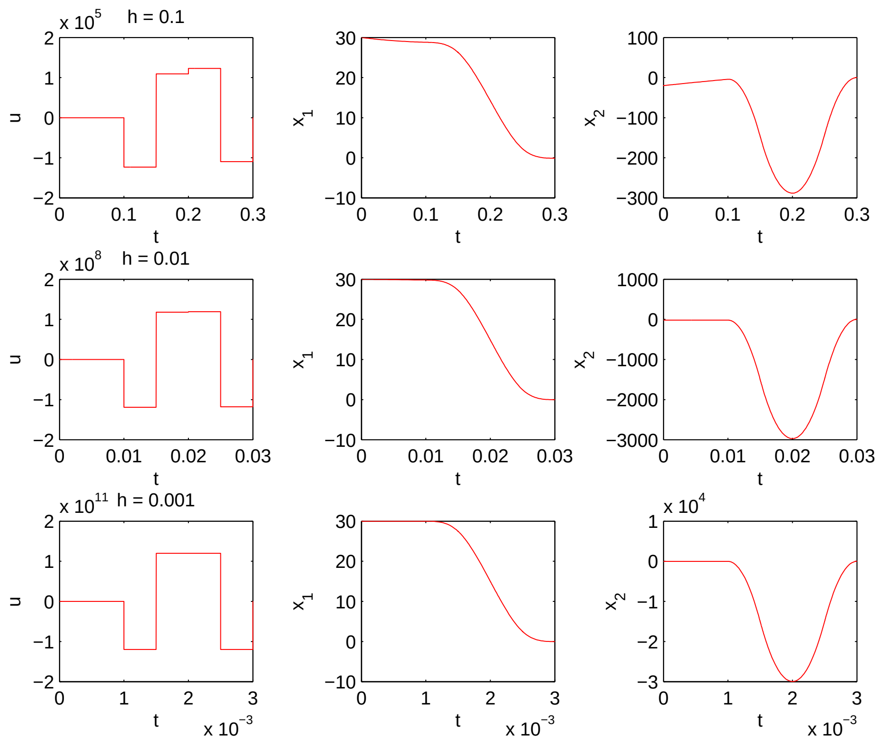


Fig. 4: Input and state response using the impulsive input approach with piecewise constant approximation for different times h.

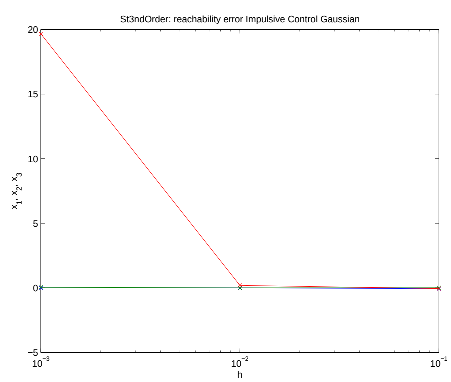


Fig. 5: State value after applying the impulsive input approach with Gaussian approximation for different times h.

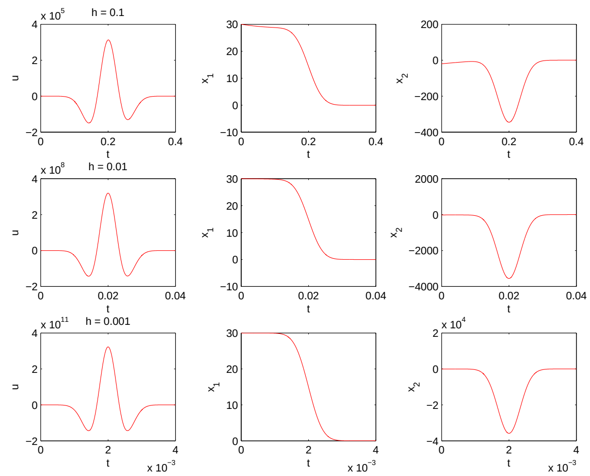


Fig. 6: Input and state response using the impulsive input approach with Gaussian approximation for different times h.

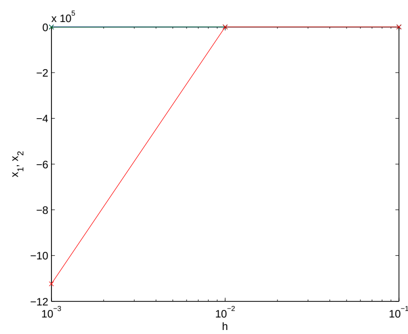


Fig. 7: State value after applying the nulling input for different times h.

$$W_h = \exp(Ah)\tilde{W}_h \exp(A^T h).$$

Therefore, inverting W_h is just as difficult as inverting $\tilde{W}_h$. Figure 9 represents the condition number of $\tilde{W}_h$, that is defined as the ratio of the largest and the smallest singular value. It is well known that a large value of the condition number is indicating that the matrix is badly conditioned numerically, and it is easy to see that for $h = 0.01$, the condition number is 10^{10}, whereas for $h = 0.001$, the condition number is around 10^{15}. This explains the failure of the direct approach for these values of h, whereas the impulsive input approach is clearly not affected by this issue.

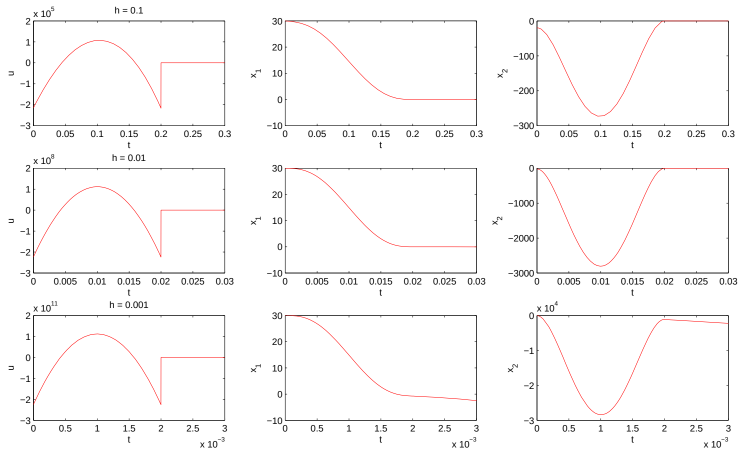


Fig. 8: Input and state response using the direct approach.

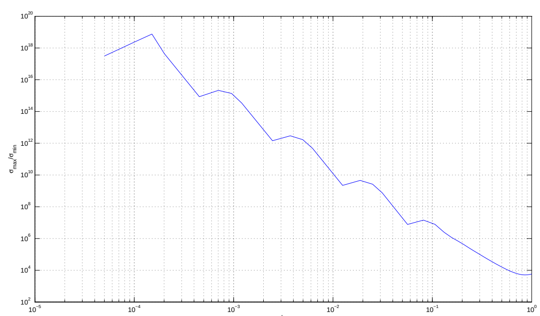


Fig. 9: Numerical condition number of $\tilde{W}_h$ as a function of h.

The phenomenon illustrated in this example is generic. Even in the two dimensional case, we show that the condition number of W_h tends to infinity as h decreases to zero. Indeed let

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

It is easy to compute the expression of W_h in this case explicitly as

$$\begin{aligned} W_h &= \begin{bmatrix} w_1(h) & w_{12}(h) \\ w_{12}(h) & w_2(h) \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{2a_1 h}-1}{2a_1} b_1^2 & \frac{e^{(a_1+a_2)h}-1}{a_1+a_2} b_1 b_2 \\ \frac{e^{(a_1+a_2)h}-1}{a_1+a_2} b_1 b_2 & \frac{e^{2a_2 h}-1}{2a_2} b_2^2 \end{bmatrix}, \end{aligned} \tag{15}$$

where can see that, as $h \to 0$

$$\frac{w_1(h)}{h} \to b_1^2,\ \frac{w_2(h)}{h} \to b_2^2,\ \frac{w_{12}(h)}{h} \to b_1 b_2.$$

The two eigenvalues of W_h are

$$\lambda_1(h) = \frac{1}{2}(w_1 + w_2 + \sqrt{(w_1 - w_2)^2 + 4w_{12}^2}),$$
$$\lambda_2(h) = \frac{1}{2}(w_1 + w_2 - \sqrt{(w_1 - w_2)^2 + 4w_{12}^2}).$$

Using repeatedly the fact that $\lim_{h\to 0} \frac{e^{ah}-1}{ah} = 1$, and the expressions in (15), we readily deduce that

$$\lim_{h\to 0} \frac{\lambda_1(h)}{h} = b_1^2 + b_2^2,$$

which is not zero unless the system is uncontrollable. On the other hand

$$\lim_{h\to 0} \frac{\lambda_2(h)}{h} = 0.$$

Consequently

$$\lim_{h\to 0} \frac{\lambda_1(h)}{\lambda_2(h)} = \infty,$$

which shows that W_h becomes badly conditioned as h becomes small. It is very likely that this result holds true for the higher dimensional case, but it is already clear that the example presented in this section is not isolated in the sense that the impulsive control approach is better suited than the direct approach to drive the state quickly to the origin.

5 Application to an orbital rendezvous problem

We consider the linearized model for orbital rendezvous that is well-known as the Clohessy-Wiltshire equations [14] that expresses the relative motion of a chasing spacecraft in the coordinate system fixed to the target spacecraft, as represented in Figure 10,

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} - 3\omega^2 x &= u_x, \\ \ddot{y} + 2\omega\dot{x} &= u_y, \\ \ddot{z} + \omega^2 z &= u_z, \end{aligned} \tag{16}$$

where ω is the orbital rate, x, y and z are the components of the relative displacement between chasing spacecraft and the target, and u_x, u_y and u_z are the components of the thrust acceleration of the chasing spacecraft. If we denote by $X = \begin{bmatrix} x \; y \; z \; \dot{x} \; \dot{y} \; \dot{z} \end{bmatrix}^T$ the state space vector of this model, and by $U = \begin{bmatrix} u_x \; u_y \; u_z \end{bmatrix}^T$ the input vector, the motion equations (16) can be written as

$$\frac{d}{dt}X = AX + BU, \tag{17}$$

where

$$A = \begin{bmatrix} & O_3 & & & I_3 & \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} O_3 \\ I_3 \end{bmatrix}.$$

Clearly, the rendezvous problem can be formulated as bringing the initial state $X(t_o)$ to a final state $X(t_f) = \begin{bmatrix} x_f \; y_f \; z_f \; 0 \; 0 \; 0 \end{bmatrix}^T$, where x_f, y_f and z_f are the final relative displacements between the two spacecrafts. We seek inputs of the form

$$U(t) = \delta(t - t_i)\alpha_o + \delta'(t - t_i)\alpha_1,$$

where α_o and α_1 are constant vectors in $\mathbb{R}^3$. The state at final time t_f is

$$X(t_f) = e^{A(t_f - t_o)}X(t_o) + e^{A(t_f - t_i)}B\alpha_o + e^{A(t_f - t_i)}AB\alpha_1.$$

If the initial and the final states are known, this relation can readily be solved for the impulse coefficients

$$\begin{bmatrix} \alpha_1 \\ \alpha_o \end{bmatrix} = \begin{bmatrix} AB \; B \end{bmatrix}^{-1} [e^{A(t_i - t_f)}X(t_f) - e^{A(t_i - t_o)}X(t_o)].$$

This expression can be used to give an analytic expression for the impulse coefficients if we notice that

$$\begin{bmatrix} AB \; B \end{bmatrix}^{-1} = \begin{bmatrix} I_3 & O_3 \\ -A_o & I_3 \end{bmatrix},$$

with $A_o = \begin{bmatrix} 0 & 2\omega & 0 \\ -2\omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and that

$$e^{A\tau} = \begin{bmatrix} 2 - \cos\omega\tau & 0 & 0 & \frac{\sin\omega\tau}{\omega} & \frac{2(1-\cos\omega\tau)}{\omega} & 0 \\ 6(\sin\omega\tau - \omega\tau) & 1 & 0 & \frac{2(\cos\omega\tau - 1)}{\omega} & \frac{4\sin\omega\tau - 3\omega\tau}{\omega} & 0 \\ 0 & 0 & \cos\omega\tau & 0 & 0 & \frac{\sin\omega\tau}{\omega} \\ 3\omega\sin\omega\tau & 0 & 0 & \cos\omega\tau & 2\sin\omega\tau & 0 \\ 6(\cos\omega\tau - 1) & 0 & 0 & -2\sin\omega\tau & 4\cos\omega\tau - 3 & 0 \\ 0 & 0 & -\omega\sin\omega\tau & 0 & 0 & \cos\omega\tau \end{bmatrix}.$$

By applying these relations and the approximations schemes proposed here, it is possible to devise efficient online algorithms for computing the steering thrust for solving the rendezvous problem. For illustration purposes, we consider a numerical example with the parameter values given in Table 1. Only the piecewise constant approximation approach is considered as it is better suited for the case of solid fuel thrusters. The time h was succesively varied from 200 s, 100 s, and 20 s. The results of the simulations are represented in Figure 11, Figure 12, and respectively Figure 13. As expected, the necessary thrust acceleration level is increasing as h becomes smaller. However, the vectors α_o and α_1 are independent of h. It is therefore easy to determine, using our approach, a minimum h that is compatible with the maximum achievable thrust acceleration.

The proposed impulsive control technique, combined with a robust feedback control, including traditional and higher order sliding mode control algorithms (see e.g. [15, 16]) can also be applied to the satellite formation control problem considered in [17]. However, such a closed loop implementation will be the subject of future work.

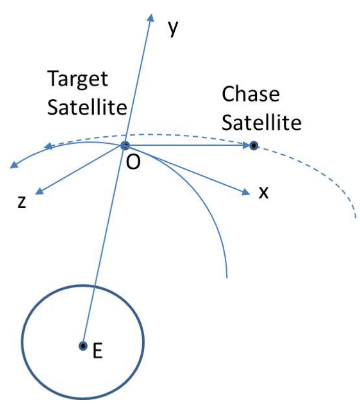
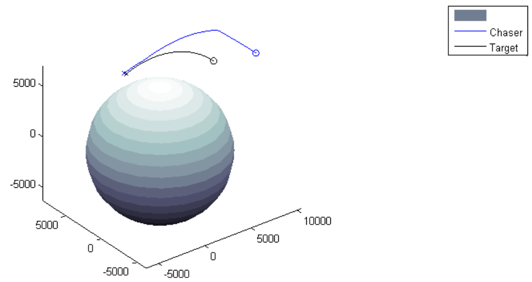


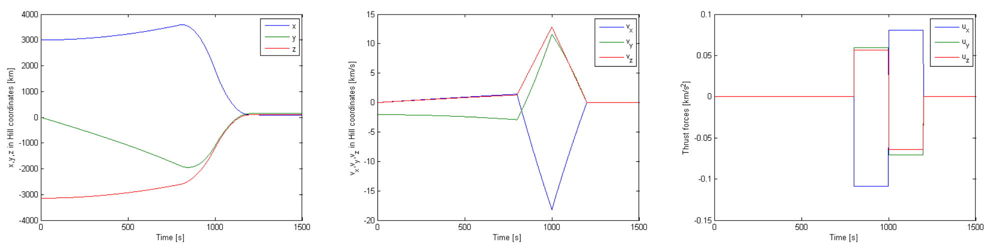
Fig. 10: Schematic representation of the satellite rendez-vous problem.

Parameter	Symbol	Value
Major semi-axis target	R_T	9000 km
Orbital inclination target		-50 deg
Orbital rate of the target	$\omega = \sqrt{\frac{\mu}{R_T^3}}$	7.39444 10^{-4} rad/s
Initial semi-axis chaser	R_I	12000 km
Orbital inclination chaser		-30 deg
Final time	t_f	1500 s
Impulse application time	t_i	1000 s
Rendez-vous position (Hill coordinates)	x_f, y_f, z_f	$10, 10, 10$ km
Time step for impulse approximation	h	200, 100, 20 s

Table 1: Numerical values of the parameters for the rendez-vous problem.



(a) Three dimensional orbits.



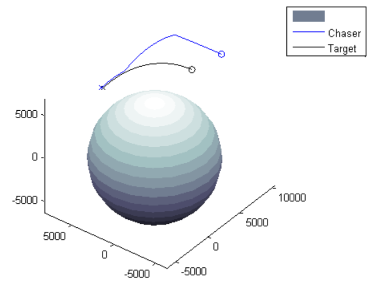
(b) Positions, velocities and thrust accelerations of chaser in Hill coordinates.

Fig. 11: Simulation results for the case of pulse width $h = 200$ s.

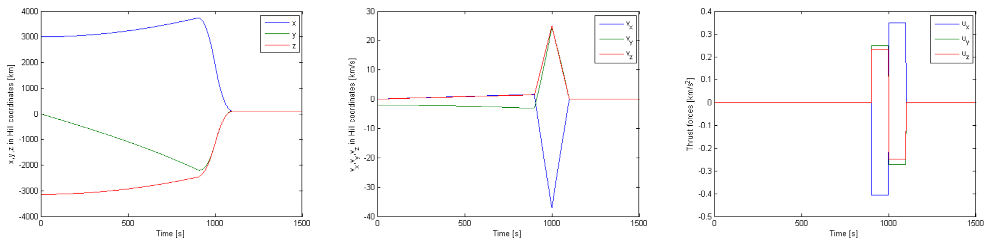
6 Conclusions and way forward

An impulsive input approach to the problem of driving the state of a linear system to the origin in very short time is studied in this work. The control input was derived as a linear combination of the Dirac delta function and its derivatives. Subsequently, two approximation schemes were proposed for approximating the impulsive input and theoretical results were proven to confirm their validity. Using a numerical example, we have shown that a direct approach to obtain a nulling input by solving an integral equation runs into numerical problems for short time intervals, whereas the solutions obtained by the impulsive input approach are not affected. For the second order case, we showed that the numerical problems are generic and not particular to the chosen example. Another observation is that the approximation using the Gauss function may give poor results due to the unbounded support, although this seems to be the approximation most studied in the literature.

Future work will concentrate on combining the proposed approach with robust feedback control, including adaptive output feedback sliding mode estimation and control.



(a) Three dimensional orbits.



(b) Positions, velocities and thrust accelerations of chaser in Hill coordinates.

Fig. 12: Simulation results for the case of pulse width $h = 200$ s.

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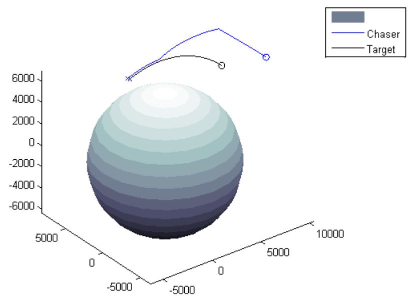
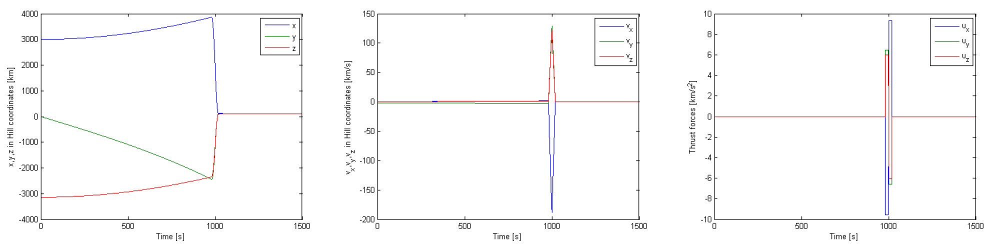


(a) Three dimensional orbits.



(b) Positions, velocities and thrust accelerations of chaser in Hill coordinates.

Fig. 13: Simulation results for the case of pulse width $h = 20$ s.

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