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Similarities of Hedging and $\mathcal{L}_1$ Adaptive Control

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Abstract In recent years $\mathcal{L}_1$ adaptive control was suggested as an advancement to model reference adaptive control (MRAC) and its benefits have been controversially discussed. This paper discusses the similarities of applying a hedging signal to the reference model used in model reference adaptive control to account for dynamic constraints in the input channel, and $\mathcal{L}_1$ adaptive control. In particular it is shown that in the case where the control effectiveness is known, both approaches are exactly the same, where the contribution of the $\mathcal{L}_1$ theory is the mathematically correct framework that provides a stability proof/condition which has not been available for the hedging approach. In the case of unknown control effectiveness, the two methods are slightly different and the $\mathcal{L}_1$ approach additionally adjusts the cutoff frequency of the low-pass filter. This difference allows for the elegant stability proof given by $\mathcal{L}_1$ theory. At the end the two approaches are compared based on a simple short period model of a large transport aircraft by assessing the robust performance w.r.t. model uncertainties.

1 Introduction

In the last years model reference adaptive control (MRAC) has attracted large interest in the aerospace community and the theory is part of many standard textbooks on nonlinear and adaptive control [1,2,3,4]. The approach is based on an online adjustment of the controller parameters which is driven by the demand to follow a specified reference dynamics. Hence the approach seems to be well suited to increase the robust performance in the presence of parametric uncertainties

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and failures, and it might reduce the reliance on modeling data necessary for the controller design.

Even so the enhanced interest led to major advances in the theoretical framework and the theory was applied to many practical problems, the application of adaptive flight control is still almost exclusively limited to research projects. The reason is that the certification of adaptive control still remains an open challenge, as classic, linear compliance metrics (e.g. gain and phase margin) cannot be applied to the inherently nonlinear system. Instead Lyapunov's direct method must be utilized to show stability, but it allows no intuitive and empiric conclusion on the robustness. Furthermore, the adaptive system has a time varying character, what makes the certification even more critical. In particular it remains a challenge to assess the transient response by suitable metrics during adaptation. To achieve good transient performance, fast adaptation to the correct parameters would be desired. However, it is well known in adaptive control that large adaptive gains, which are necessary for fast adaptation, reduce the robustness w.r.t. time delay [5,6], as they act as integral feedback gains. To solve this drawback the theory of $\mathcal{L}_1$ adaptive control was developed [5,7,8], with the objective to decouple robustness and adaptation and thus decouple control from estimation. To achieve this, a low-pass filter is introduced at a very particular place in the control structure with the objective to allow only input signals that are within the bandwidth of the input channel and, hence, render the control objective achievable.

This new architecture and the claims that are made within $\mathcal{L}_1$ adaptive control theory raised concerns in the adaptive control community because some researchers think that the claims cannot be met. However, in the following it is shown that $\mathcal{L}_1$ adaptive control basically provides a method to account for dynamic constraints in the input channel of the plant, legitimatized by sound mathematical proofs.

In general the idea to account for input constraints in adaptive control is not new but was already suggested in [9], however the suggested modification was introduced to account for hard nonlinearities in form of input saturation. Actuator saturation especially poses difficulties for MRAC-type controllers as it violates some of the basic assumptions. If the reference model does not account for actuator saturation this directly means that the real plant cannot follow the reference model for every command that is issued, because a linear reference model assumes unlimited control energy, whereas for the plant this is not available. In the case of saturation an error, which cannot be compensated, will result from the error dynamics (difference between plant dynamics and reference dynamics), and as the parameters are adjusted based on integration of this error they will increase as long as the input saturates. To overcome this problem the use of a hedging signal to modify the error dynamics was already suggested in [9]. In the adaptive control literature the word hedging is in general used to refer to a modification of certain signals with the objective to "hide" certain input characteristics of the plant from the error dynamics. As mentioned the approach was originally suggested to account for saturations and a stability proof for SISO systems was given in [10] and further developed in [11]. However, the method is not only suitable to account for input constraints like actuator or rate saturation, but it can also be used to account for dynamic input constraints in the input channel of the plant [12]. And as shown in the following this is in some cases equivalent to $\mathcal{L}_1$ adaptive control.

2 Adaptive Control with Known Control Effectiveness

Let the considered system dynamics be given by

$$\dot{\mathbf{x}}_P(t) = \mathbf{A}_P \cdot \mathbf{x}_P(t) + \mathbf{b}_P \cdot u(t) \tag{1}$$

where $\mathbf{x}_P \in R^n$ is the state vector of the system, $u \in R$ is the input of the system, $\mathbf{A}_P \in R^{n\times n}$ is the unknown system matrix, $\mathbf{b}_P \in R^{n\times 1}$ is the known input matrix.

The desired behavior for the system that should be achieved by the controller is specified by a linear reference model of the form

$$\dot{\mathbf{x}}_M(t) = \mathbf{A}_M \cdot \mathbf{x}_M(t) + \mathbf{b}_M \cdot r(t)\,, \tag{2}$$

where $\mathbf{x}_M \in R^n$ is the desired reference trajectory, $r \in R$ is the reference command, $\mathbf{A}_M \in R^{n\times n}$ is the desired, stable system matrix, and $\mathbf{b}_M \in R^{n\times 1}$ is the desired input matrix.

Next the following control law is chosen

$$u_{CMD}(t) = \boldsymbol{\theta}_x^T(t) \cdot \mathbf{x}_P(t) + k_r r(t)\,. \tag{3}$$

$\boldsymbol{\theta}_x^T \in R^{1\times n}$ is a parameter to compensate matched uncertainties which are linear in the states (matched uncertainties in $\mathbf{A}_P$), and it is adjusted by adaptation. Because the time dependency is usually clear from the context, in the following time dependency is not explicitly denoted to improve readability. The feedforward gain k_r is chosen such that $\mathbf{b}_M = \mathbf{b}_P k_r$.

Without robustness modification the stability proof requires all uncertainties to be matched, that means they have to be in the span of $\mathbf{b}_P$. This gets obvious by inserting the control law Eq.(3) in the plant dynamics Eq.(1) and comparing the closed loop system to the reference model Eq.(2). That means, to achieve the desired behavior the following matching condition needs to hold, where the ideal parameters, marked by superscript asterisk, need to exist so that the equations can be satisfied:

$$\mathbf{A}_M = \mathbf{A}_P + \mathbf{b}_P \boldsymbol{\theta}_x^{*T} \quad (4)$$

The stability properties mentioned in this section only hold under these conditions. Requiring the matching condition to hold is equal to assuming that the plant can be denoted by

$$\dot{\mathbf{x}}_P = \mathbf{A}_M \cdot \mathbf{x}_P + \mathbf{b}_P \cdot \left[u - \boldsymbol{\theta}_x^{*T} \mathbf{x}_P \right]. \quad (5)$$

The control law Eq.(3) guarantees global stability of the closed loop system if the following update law is used [1,2,3]

$$\dot{\boldsymbol{\theta}}_x = -\boldsymbol{\Gamma}_x \mathbf{x}_P \cdot \mathbf{e}_c^T \mathbf{P} \mathbf{b}_P , \quad (6)$$

where $\mathbf{e}_C = \mathbf{x}_P - \mathbf{x}_M$ and $\mathbf{P} = \mathbf{P}^T \in R^{n \times n}$ is the symmetric, positive definite solution of the Lyapunov equation $\mathbf{A}_M^T \mathbf{P} + \mathbf{P}\mathbf{A}_M = -\mathbf{Q}_0$, with $\mathbf{Q}_0$ being a symmetric, positive definite (design) matrix, and $\boldsymbol{\Gamma}_x \in R^{n \times n}$ is a symmetric, positive definite design parameter that determines the adaptation speed. Furthermore, Barbalat's Lemma guarantees that the error converges to zero [1,2,3].

In the following the problem is considered that additional dynamics (e.g. actuator or structural filter dynamics) are present in the input channel of the plant, and it is assumed that they can be represented by stable, strictly-proper, linear low-pass filter $C(s)$ with DC gain of 1, where s is the Laplace variable,

$$u(s) = C(s) u_{CMD}(s) \quad (7)$$

and u_{CMD} is the commanded input. These input constraints are present in any real system due to actuator dynamics. Especially in aircraft control it is state of the art to further limit the bandwidth of the input channel by introducing structural filters to avoid an excitation of the structural modes.

If we insert the control law Eq.(3) in the plant Eq.(5) with the input dynamics of Eq.(7) and build the error dynamics we obtain

$$\dot{\mathbf{e}}_c = \mathbf{A}_M \cdot \mathbf{e}_c + \mathbf{b}_P \cdot \tilde{\boldsymbol{\theta}}_x^T \cdot \mathbf{x}_P + \mathbf{b}_P \Delta u , \quad (8)$$

where $\tilde{\boldsymbol{\theta}}_x = \boldsymbol{\theta}_x - \boldsymbol{\theta}_x^*$ is the parameter error and

$$\Delta u(s) = \left(C(s) - 1 \right) u_{CMD}(s) \quad (9)$$

is the control deficiency. Here we can see that the additional dynamics in the input channel leads to an unexpected control deficiency Δu that causes an additional term in the error dynamics. This introduces an excitation of the error dynamics, and to guarantee stability it has to be either accounted for by a robustness modification of the adaptive law or by hedging as shown in the next section.

2.1 MRAC with Hedging

To hide the effect of the control deficiency on the error dynamics a hedging signal can be used. That means either the error signals are augmented directly, as suggested in [9,10], or indirectly, as shown in [9,12], by a modification of the reference dynamics. That means to remove the effect of the input dynamics we have two options, where for the first approach we can define an additional differential equation

$$\Delta\dot{\mathbf{e}}_c = \mathbf{A}_M \cdot \Delta\mathbf{e}_c + \mathbf{b}_P \Delta u \tag{10}$$

and augment the error signal by $\mathbf{e}_U = \mathbf{e}_C - \Delta\mathbf{e}_C$. Now in the error dynamics of $\mathbf{e}_U$ the excitation due to the control deficiency is removed and one obtains

$$\dot{\mathbf{e}}_U = \mathbf{A}_M \cdot \mathbf{e}_U + \mathbf{b}_P \cdot \tilde{\boldsymbol{\theta}}_x^T \cdot \mathbf{x}_P \,. \tag{11}$$

The second approach, which is for example suggested in [12], is to directly modify the reference dynamics with the hedging signal of Eq.(9) in the form

$$\dot{\mathbf{x}}_M = \mathbf{A}_M \cdot \mathbf{x}_M + \mathbf{b}_M \cdot r + \mathbf{b}_P \{(C(s)-1)u_{CMD}(s)\}_t \,. \tag{12}$$

Here $\{\bullet\}_t$ denotes that the term inside the brackets is transformed from the Laplace domain to the time domain and in the following $\{\bullet\}_s$ denotes the reverse transformation.

In figure 1 the structure of the closed loop MRAC with hedging is shown. The augmentation of the reference model leads to the same error dynamic as shown in Eq.(11), and thus the two approaches are equivalent.

Using $\mathbf{e}_U$ instead of $\mathbf{e}_C$ in the update law Eq.(6) guarantees global asymptotic stability of the error dynamics. However, due to the feedback of Δu to the reference model it is not a priori guaranteed that the reference model is stable. Here the $\mathcal{L}_1$ theory provides a stability condition, which is given in the next section.

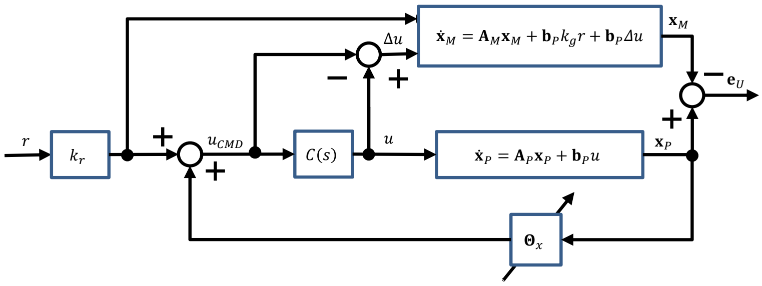


Fig. 1 Architecture of MRAC with hedging

2.2 $\mathcal{L}_1$ Adaptive Control

For $\mathcal{L}_1$ adaptive control a predictor based approach is used and the state predictor is given by [5,7,8]

$$\dot{\hat{\mathbf{x}}}_P = \mathbf{A}_M \cdot \hat{\mathbf{x}}_P + \mathbf{b}_P \left[u - \boldsymbol{\theta}_x^T \mathbf{x}_P \right], \tag{13}$$

where u is the low-pass filtered input. The structure of the $\mathcal{L}_1$ adaptive control is shown in figure 2 and in the following it is shown that the predictor is equal to the reference model with hedging.

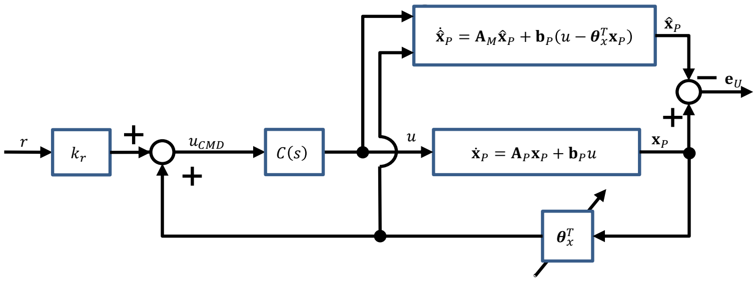


Fig. 2 Architecture of $\mathcal{L}_1$ adaptive control

Inserting Eq.(7) and adding and subtracting $\mathbf{b}_P k_r r$ we get

$$\dot{\hat{\mathbf{x}}}_P = \mathbf{A}_M \cdot \hat{\mathbf{x}}_P + \mathbf{b}_P \left[\{C(s) u_{CMD}(s)\}_t - \boldsymbol{\theta}_x^T \mathbf{x}_P - k_r r + k_r r \right]. \tag{14}$$

With Eq.(3) we obtain from Eq.(14)

$$\dot{\hat{\mathbf{x}}}_P = \mathbf{A}_M \cdot \hat{\mathbf{x}}_P + \mathbf{b}_P k_r r + \mathbf{b}_P \{(C(s)-1)u_{CMD}(s)\}_t \,. \tag{15}$$

Clearly this shows that in the presence of input dynamics Eq.(7) the predictor from Eq.(13) is mathematically equal to the reference model with hedging of Eq.(12), and the only difference can be found in the structural implementation. Thus it is clear that both approaches have the same properties and result in the same performance.

However, an important improvement is that the theory of $\mathcal{L}_1$ adaptive control provides a stability condition for the state predictor that also guarantees stability for the reference model modified by hedging. Basically the stability condition results from showing BIBO stability by application of the Small Gain Theorem. And by using the $\mathcal{L}_\infty$-norm the following $\mathcal{L}_1$-norm condition is induced [5]:

$$\|\mathbf{G}(s)\|_{\mathcal{L}_1} \cdot \alpha < 1 \tag{16}$$

With $\|\mathbf{G}(s)\|_{\mathcal{L}_1} = \max_{i=1,\dots,n} \|G_i(s)\|_{\mathcal{L}_1}$, $\|G_i(s)\|_{\mathcal{L}_1}$ being defined as the $\mathcal{L}_1$-norm of the impulse response $\|g_i(t)\|_{\mathcal{L}_1}$, $\mathbf{G}(s) = (s\mathbf{I} - \mathbf{A}_M)^{-1}\mathbf{b}_P(C(s)-1)$, and

$$\alpha = \max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \sum_{i=1.}^{n} |\theta_i| \,. \tag{17}$$

Here it is furthermore assumed that the uncertainty $\boldsymbol{\theta}$ is limited to a convex, compact set by means of projection and that $\boldsymbol{\theta}^*$ also lies inside this set.

3 Adaptive Control with Unknown Control Effectiveness

In difference to section 2 it is now assumed that there is an additional uncertainty in the control effectiveness λ and it is assumed that the sign of the control effectiveness is known $\lambda \in R^+$. So similar to Eq.(5) the following parameterization for the plant is assumed

$$\dot{\mathbf{x}}_P = \mathbf{A}_M \cdot \mathbf{x}_P + \mathbf{b}_P \cdot \left[\lambda u - \boldsymbol{\theta}_x^{*T} \mathbf{x}_P\right] \tag{18}$$

and again it is assumed that additional dynamics are present in the input channel so that u is given by Eq.(7).

3.1 MRAC with Hedging

To account for the additional uncertainty the following control law is applied

$$u_{CMD} = \hat{\lambda}^{-1}\left[\boldsymbol{\theta}_x^T \cdot \mathbf{x}_P + k_r r\right], \tag{19}$$

where $\hat{\lambda}$ is an estimation of λ. This estimation is obtained by the update law

$$\dot{\hat{\lambda}} = \gamma_\lambda u \cdot \mathbf{e}_c^T \mathbf{P}\mathbf{b}_P , \tag{20}$$

where $\gamma_\lambda \in R^+$ is a positive design parameter. For the control law in Eq.(19) it is clear that $\hat{\lambda}$ must be bounded away from zero by projection.

Similar to the previous example the control deficiency due to the input dynamics can be accounted for by hedging and then the reference model is given by

$$\dot{\mathbf{x}}_M = \mathbf{A}_M \cdot \mathbf{x}_M + \mathbf{b}_M \cdot r + \mathbf{b}_P \hat{\lambda} \Delta u . \tag{21}$$

Or equivalently the following predictor can be used (as illustrated in the previous section 2.2)

$$\dot{\hat{\mathbf{x}}}_P = \mathbf{A}_M \cdot \hat{\mathbf{x}}_P + \mathbf{b}_P \cdot \left[\hat{\lambda} u - \boldsymbol{\theta}_x \mathbf{x}_P\right]. \tag{22}$$

When Eq.(7) is inserted in Eq.(22) we obtain

$$\dot{\hat{\mathbf{x}}}_P = \mathbf{A}_M \cdot \hat{\mathbf{x}}_P + \mathbf{b}_M r + \mathbf{b}_P \cdot \left[\left\{\hat{\lambda}(s) * C(s)\left\{\hat{\lambda}^{-1}\left[\boldsymbol{\theta}_x \cdot \mathbf{x}_P + k_r r\right]\right\}_s\right\}_t\right]. \tag{23}$$

Due to the time varying $\hat{\lambda}$ the derivation of a stability condition becomes more conservative than the one that led to the stability requirement of section 2.2. Furthermore, for $\mathcal{L}_1$ adaptive control a slightly different control approach is chosen, which is shown in the following section and this results in a less conservative stability condition and performance bounds [5]. However, in analogy to Eq.(16) and by following the derivation in [5], for the current problem the following condition can be derived to guarantee stability of the reference model of Eq.(21)

$$\left\|\mathbf{H}(s)\right\|_{\mathcal{L}_1} \left\|C(s) - 1\right\|_{\mathcal{L}_1} \frac{\hat{\lambda}_{\max}}{\hat{\lambda}_{\min}} \cdot \alpha < 1 \tag{24}$$

$$\mathbf{H}(s) = (s\mathbf{I} - \mathbf{A}_M)^{-1} \mathbf{b}_P . \tag{25}$$

The condition is more conservative because the transfer functions $\mathbf{H}(s)$ and ($C(s)-1$) cannot be multiplied in the frequency domain because of the time varying $\hat{\lambda}$. Thus the norm of $\|\mathbf{H}(s)\|_{\mathcal{L}_1}$ and $\|C(s)-1\|_{\mathcal{L}_1}$ must be taken separately, leading to a more conservative stability condition. Here it is assumed that $\hat{\lambda}$ is limited by projection, which means that $\hat{\lambda}$ is constrained to a convex set such that $\hat{\lambda} \in \left[\hat{\lambda}_{\min}, \hat{\lambda}_{\max}\right]$, and the projection algorithm assures that the time course of the parameters is $\in C^1$ [5].

3.2 $\mathcal{L}_1$ Adaptive Control

For $\mathcal{L}_1$ adaptive control the same predictor Eq.(22) and the same update laws Eq.(6) and Eq.(20) are used, however a different control law than in MRAC is used and given by

$$u_{\mathcal{L}_1}(s) = -kD(s) \cdot \left\{\hat{\lambda} u_{\mathcal{L}_1} - \boldsymbol{\theta}_x^T \cdot \mathbf{x}_P - k_r r\right\}_s . \tag{26}$$

Or equivalently this control law can be denoted by

$$u_{\mathcal{L}_1}(s) = -kD(s) \cdot \left\{\hat{\lambda}\left(u_{\mathcal{L}_1} - \frac{1}{\hat{\lambda}}\left(\boldsymbol{\theta}_x^T \cdot \mathbf{x}_P + k_r r\right)\right)\right\}_s , \tag{27}$$

where $k > 0$ and $D(s)$ is a strictly proper transfer function. If $\hat{\lambda}$ would be constant and equal to the true value λ this would result in

$$u_{\mathcal{L}_1}(s) = \frac{\lambda k D(s)}{1+\lambda k D(s)} \left\{\frac{1}{\lambda}\left(\boldsymbol{\theta}_x^T \cdot \mathbf{x}_P + k_r r\right)\right\}_s = C^*(s)\left\{\frac{1}{\lambda}\left(\boldsymbol{\theta}_x^T \cdot \mathbf{x}_P + k_r r\right)\right\}_s , \tag{28}$$

where $C^*(s)$ is a stable low-pass filter whose cutoff frequency depends on λ. That means Eq.(26) basically implements a low-pass filter where the cutoff frequency is adjusted by the estimation of $\hat{\lambda}$. This also gets obvious in figure 3 where the two different implementations of Eq.(26) and Eq.(27) are shown.

So the main difference between the hedging approach and $\mathcal{L}_1$ adaptive control is that the cutoff frequency of the low-pass filter in $\mathcal{L}_1$ adaptive control is adjusted by the estimated control effectiveness. The hedging was motivated by removing the excitation of the error dynamics caused by a control deficiency where the $\mathcal{L}_1$

approach is chosen to derive certain performance bounds [5] in an elegant way. However if the input dynamic is already fixed, as for example by an actuator, the $\mathcal{L}_1$ approach from section 3.2 cannot be applied as the dynamics of the actuator cannot be adjusted by $\hat{\lambda}$. Of course one can always add an additional filter according to Eq.(26) to further restrict the bandwidth in the input channel.

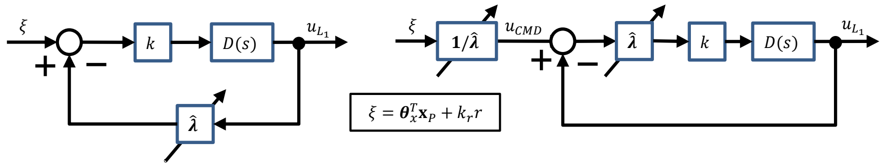


Fig. 3 Different implementations of the adaptive filter

4 Simulation Results

The following model is based on the linearized short period dynamics of a large transport aircraft and includes a nonlinear pitch-break. Pitch-break is a phenomenon where the pitch stiffness decreases with increasing angle of attack. Hence, the system becomes less stable, resulting in degraded handling qualities.

The system equations of the short-period dynamics with pitch-break nonlinearity are given by

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1+Z_q \\ M_\alpha & M_q \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} Z_\eta \\ M_\eta \end{bmatrix} \eta + \begin{bmatrix} 0 \\ f(\alpha) \end{bmatrix}, \tag{29}$$

where the states of the system are the angle of attack α in radians, which is a variation w.r.t. the trim value and the pitch rate q in radians/second, that means the state vector is $\mathbf{x}_P = [\alpha \quad q]$. η is the elevator deflection in radians, and the output available for feedback are the load factor n_Z and pitch rate q in radians, and thus

$$\begin{bmatrix} n_Z \\ q \end{bmatrix} = \begin{bmatrix} -\frac{V}{g} Z_\alpha & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} -\frac{V}{g} Z_\eta \\ 0 \end{bmatrix} \eta \,. \tag{30}$$

The coefficients of the system matrix $\mathbf{A}_P$ and the input vector $\mathbf{b}_P$ are given in table 1, $V = 253.7$ m/s, and $g = 9.81$.

Table 1 Coefficients of $\mathbf{A}_P$ and $\mathbf{b}_P$

Z_α	Z_q	M_α	M_q	Z_η	M_η
-0.704	0	-0.484	-0.654	-1.423	-77.38

The pitch up nonlinearity $f(\alpha)$ is determined by

$$f(\alpha)=\begin{cases} 0 & \text{for } \alpha\cdot 180/\pi < 1.5\,\text{deg} \\ -2M_\alpha(\alpha\cdot 180/\pi - 1.5\,\text{deg}) & \text{for } \alpha\cdot 180/\pi \geq 1.5\,\text{deg} \end{cases}. \tag{31}$$

The aircraft model furthermore contains an actuator model, a model for the computer delay, and a structural filter as shown in figure 4 and the transfer functions of the respective dynamics are given in table 2.

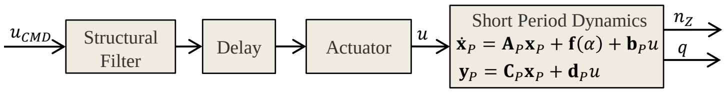


Fig. 4 Plant dynamics

Table 2 Transfer functions of the dynamics in the input channel

Actuator	**Structural Filter**	**Delay**
$\frac{5s+19800}{s^3+81s^2+3564s+19800}$	$\frac{1}{0.006s^2+0.113s+1}$	$\frac{5.21\cdot 10^{-5}s^2-0.013s+1}{5.21\cdot 10^{-5}s^2+0.013s+1}$

For the nominal plant dynamics a baseline PI controller is provided and given by

$$u_{BL}=\mathbf{k}_y^T\mathbf{y}^*+k_r r\,, \tag{32}$$

where r is the load factor reference command and $\mathbf{y}^{*T}=[n_Z \quad q \quad e_I]$, with the integrated error $e_I=\int(r-n_Z)dt$. $\mathbf{k}_y^T$ is the feedback gain matrix and k_r is the feed forward gain.

In the following, the approaches from section 3.1 and 3.2 are applied and compared. The adaptive controllers are chosen to employ the same structure as the baseline control law and thus only a linear regressor vector is used, and the adaptive feedback is provided by

$$u_{AD,fb} = \boldsymbol{\theta}_y^T \mathbf{y}^* = \begin{bmatrix}\theta_{n_Z} & \theta_q & \theta_{e_I}\end{bmatrix} \cdot \begin{bmatrix}n_Z & q & e_I\end{bmatrix}^T . \tag{33}$$

Here instead of the states $\mathbf{x}_P$ the outputs $\mathbf{y}$ are used, and thus the objective is that $\mathbf{y}^*$ tracks the reference trajectory $\mathbf{y}_M^*$, which is obtained by implementing the nominal closed loop given by Eq.(29) without the pitch-break nonlinearity, Eq.(30), and Eq.(32). This is no problem, if the output matrix has full rank and the direct feedthrough from the input to the output is zero. Although the feedthrough is small, the second condition is not satisfied for the current example as can be seen in Eq.(30). Clearly, effects like these limit the speed of adaptation, however, as the effect is very small it is treated as a disturbance here.

It should be noted that from the problem formulation it is clear that using a linear regressor violates the basic theory, because there can be no constant parameters that satisfy the matching condition and thus the parameters must be time varying.

The baseline control law is augmented with an adaptive controller that utilizes the same structure

$$u_{CMD} = \frac{1}{\hat{\lambda}}\left(u_{BL} + u_{AD,fb}\right). \tag{34}$$

For the MRAC approach with hedging the adaptive controller is composed of Eq.(34), Eq.(33) and the adaptation given by Eq.(6), and Eq.(20). An additional σ -modification term [13] was added to improve the robustness, where $\sigma = 0.01$ was chosen:

$$\dot{\boldsymbol{\theta}}_y = -\boldsymbol{\Gamma}_y\left(\mathbf{y}^* \cdot \mathbf{e}_c^T \mathbf{P}\mathbf{b}_P + \sigma \cdot (\boldsymbol{\theta}_y - \boldsymbol{\theta}_{y,0})\right), \quad \boldsymbol{\theta}_{y,0} = \mathbf{0}$$

$$\dot{\hat{\lambda}} = \gamma_\lambda\left(u \cdot \mathbf{e}_c^T \mathbf{P}\mathbf{b}_P - \sigma \cdot (\hat{\lambda} - \hat{\lambda}_0)\right), \quad \hat{\lambda}_0 = 1,$$

where $\mathbf{e}_C = \mathbf{y}^* - \mathbf{y}_M^*$. So the parameter that have to be chosen are $\boldsymbol{\Gamma}_y$ and γ_λ. To obtain $\mathbf{P}$ form the solution of the Lyapunov equation, $\mathbf{Q}_0$ was chosen to be the identity matrix. For the $\mathcal{L}_1$ adaptive controller an additional low-pass filter according to Eq.(27) is inserted in the input channel with $D(s) = 1/s$ and the additional parameter k that has to be chosen to determine the cutoff frequency. Note that $D(s)$ is only the open loop transfer function of the filter, and since the filter loop is closed (see figure3) the filter pole is stable. The adaptive parameters are limited by projection to $\boldsymbol{\theta}_{y,\min}^T = -\begin{bmatrix}6.5 & 3 & 9\end{bmatrix}$, $\boldsymbol{\theta}_{y,\max}^T = \begin{bmatrix}6.5 & 3 & 9\end{bmatrix}$, and $\hat{\lambda} \in [0.3,5]$.

The parameters of the adaptive controller are tuned with the genetic algorithm provided by Matlab©, based on a certain command signal. The cost function J for the tuning is based on a combination of weighted performance metrics, which are the $\mathcal{L}_\infty$ norm of the tracking error, $\mathcal{L}_2$ norm of the tracking error and the $\mathcal{L}_2$ norm of difference between ideal control signal and the adaptive control signal. As an additional constraint for the tuning the controllers have to provide a time delay margin of at least 0.3 seconds. The time delay margin is the amount of time delay in the input channel that brings the system to the edge of stability, and it is an often suggested robustness metric for adaptive controllers in order to replace the phase margin used in linear control theory. Due to space limitations the parameter tuning cannot be explained in more detail.

From the tuning the parameters shown in table 3 were obtained, where for $\mathcal{L}_1$ a slightly smaller value of the cost function could be obtained, but in general very similar parameters where obtained. It can be seen that in both cases quite large adaptive gains are obtained.

Table 3 Adaptive controller parameter

	$\mathbf{\Gamma}_y$	γ_λ	σ	k	J
MRAC	$\begin{bmatrix} 225 & 0 & 0 \\ 0 & 285 & 0 \\ 0 & 0 & 265 \end{bmatrix}$	1.4	0.01	-	0.245
$\mathcal{L}_1$	$\begin{bmatrix} 270 & 0 & 0 \\ 0 & 274 & 0 \\ 0 & 0 & 254 \end{bmatrix}$	1.5	0.01	22.6	0.230

For the considered case the time delay margins are given in table 4 and it is obvious that both controllers significantly reduce the time delay margin. However, this tradeoff is well known for adaptive control.

Table 4 Time delay margin

Baseline controller	**MRAC controller**	**$\mathcal{L}_1$ controller**
1.19s	0.31s	0.31s

The response for the command signal used for the gain design is shown in figure 5 and in figure 6 the error w.r.t. the reference trajectories is shown. Here it is obvious that in comparison to the baseline controller the overshoot caused by the pitch break is largely reduced and both approaches achieve almost perfect following of the reference response, and cannot be distinguished. In figure 7 the elevator deflection and rate is shown and in figure 8 and 9 the time histories for the adaptive parameters of MRAC and $\mathcal{L}_1$ are displayed.

As a manned aircraft is considered the handling qualities are of utmost importance. That means, the control laws must not only provide robust stability, but robust performance is the key property of the control law to make the aircraft controllable for the pilot. This also means it is not enough to evaluate the performance only for the considered pitch break problem, but the control law has to provide robust performance w.r.t. all kinds of uncertainties that must be expected. Therefore the performance of the adaptive controllers w.r.t. a more general set of uncertainties is assessed in the following.

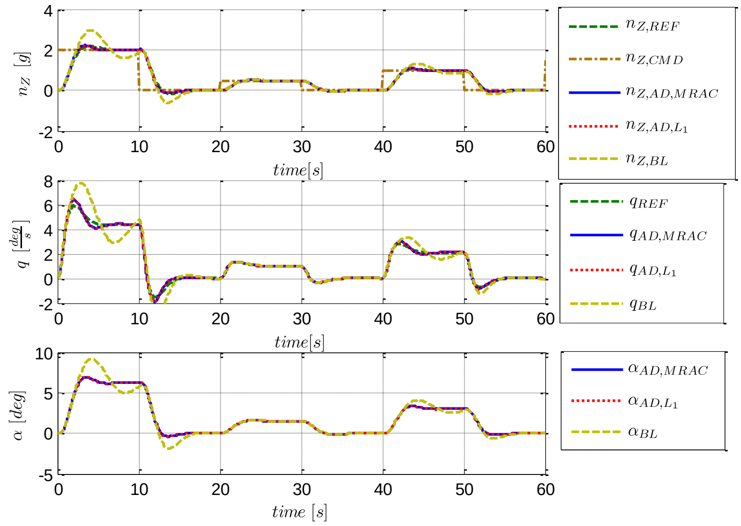


Fig. 5 Load factor and pitch rate response

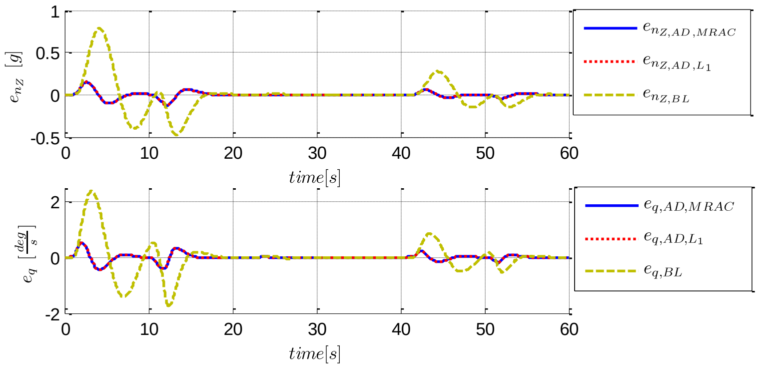


Fig. 6 Error in load factor and pitch rate response

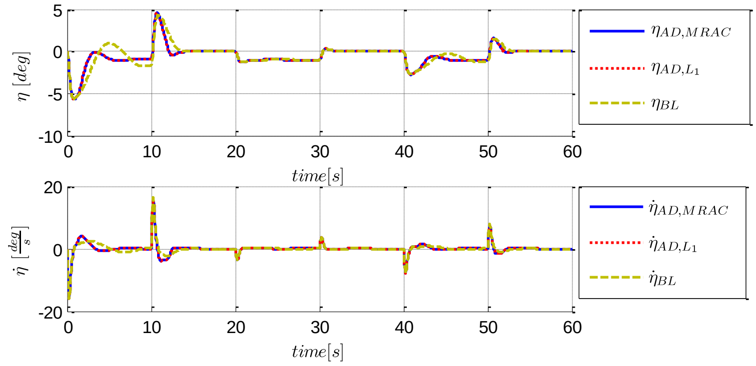


Fig. 7 Elevator deflection and rate

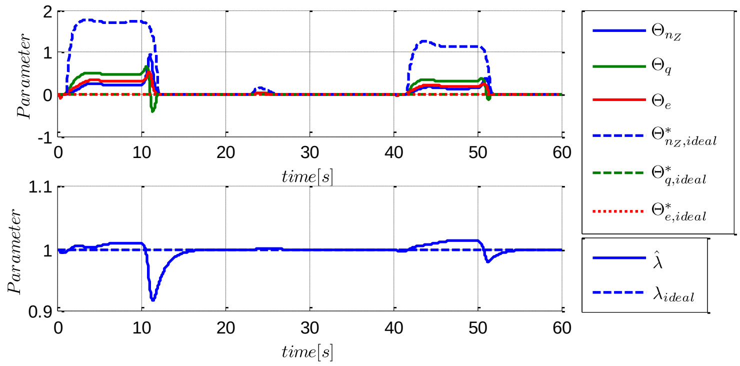


Fig. 8 MRAC adaptive controller parameters

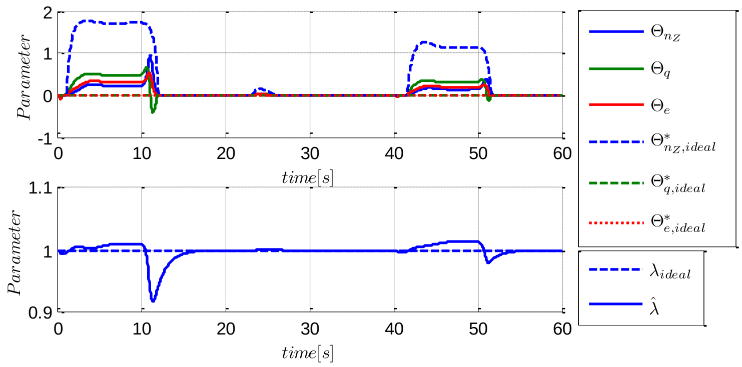


Fig. 9 $\mathcal{L}_1$ Adaptive controller parameters

To provide a measure of how a pilot perceives the adaptive and the baseline control law in presence of the nonlinearity, the frequency response function (FRF) from the input command $n_{Z,CMD}$ to the actually achieved n_Z are computed and compared for different control setups.

The FRFs were obtained from spectral estimates that were computed using the modified periodogram algorithm, as implemented in the identification software CIFER© [14]. The systems were excited in the significant frequency range by a chirp signal with an amplitude of two g's, which is shown in figure 10. This approach is based on linearity assumptions of the underlying system. Applied to nonlinear systems, the frequency response functions can be interpreted as describing functions relating the input spectrum to the output spectrum. To measure the degree of linearity, the coherence is computed.

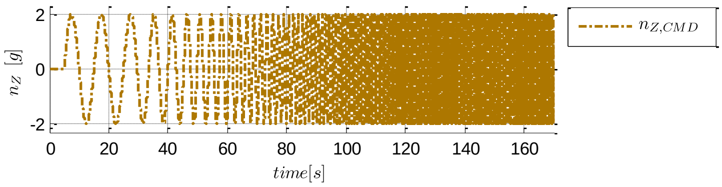


Fig. 10 Chirp signal for identification of FRFs

In figure 11 the deviations of the FRFs between the desired linear reference system are shown, on the one hand for the baseline controller applied to the nonlinear system (red line), and on the other hand for the adaptive controller applied to the nonlinear system (green line). The deviations are compared to the MUAD (maximum unnoticeable added dynamics) boundary curves, taken from [15]. The MUAD bounds are a measure of the influence of FRF deviations on the handling quality assessment of test pilots: if the FRFs of two aircraft systems differ only within the MUAD bounds, pilots will give them the same handling quality rating [16]. It can be seen that the adaptive controller significantly reduces the deviations from the reference dynamics, both in magnitude and phase. From the MUAD plot it can be followed that the adaptive controller improves the response for the considered problem as the deviation w.r.t. the reference system is almost zero over the complete frequency domain of interest. The deviation of the baseline controller also remains within the boundaries and according to this the response is still acceptable for the pilot. However, in the frequency range of the short period dynamics a clear deviation, and hence a deterioration, can be seen. In the coherence plot, we can see that with the adaptive controller the closed loop behaves almost as linear as the reference system. This can be attributed to the fast adaptation and underlines the similar dynamics of the adaptive controller and the baseline system.

According to the pitch-break problem stated above the objective of the adaptive augmentation will be to improve the response to load factor commands in the presence of the nonlinearity. Therefore boundaries for the load factor step response are defined based on the parameters: maximum overshoot, 80% rise time, 5% settling time, and 1% settling time. Based on these parameters three different boundaries are defined associated with three different levels of handling qualities (HQ) and the associated parameters given in table 5.

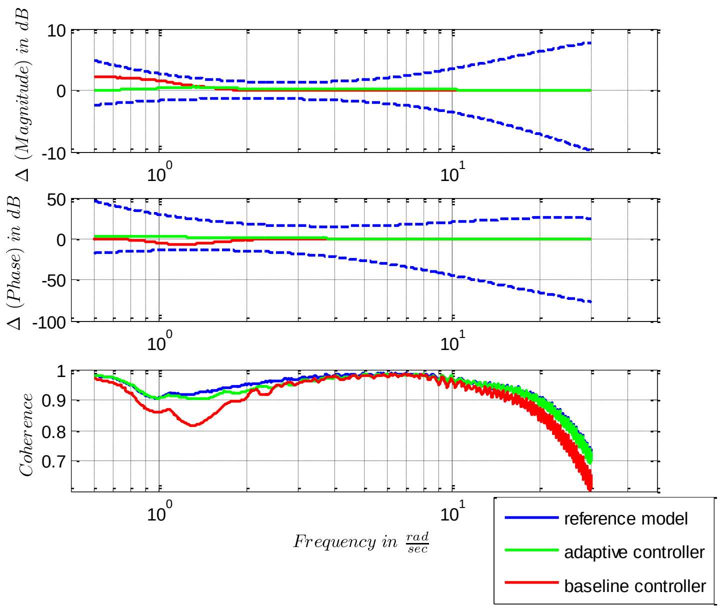
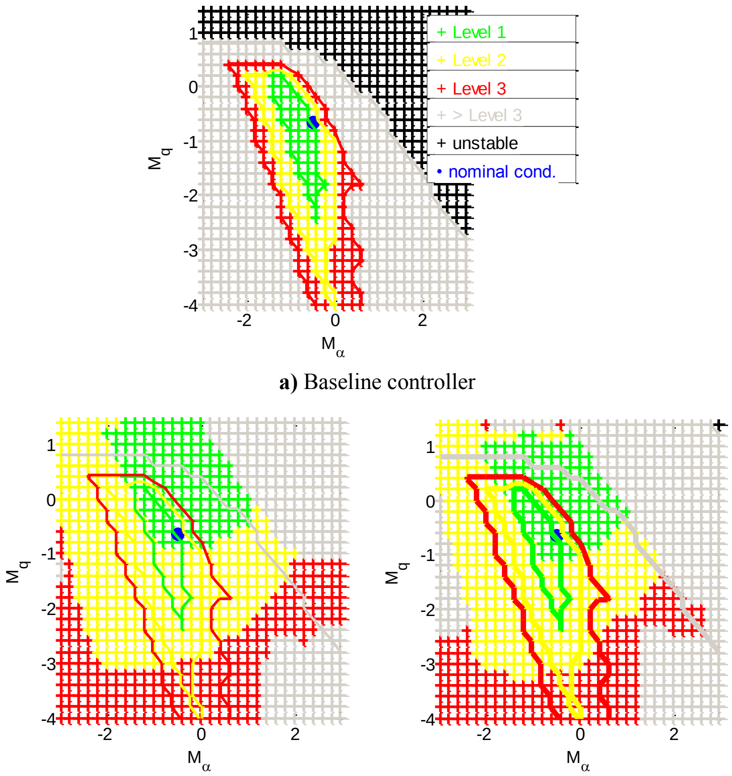


Fig. 11 $\mathcal{L}_1$ Adaptive controller parameters

The robust performance of the augmenting control laws is evaluated based on a step input together with the mentioned requirements for different kinds of uncertainties in the linear plant model and the results are compared to the performance of the baseline control law. At first the performance is evaluated over a grid of uncertainties in the coefficients determining the pitch stiffness M_α and the pitch damping M_q, which can be considered as matched or affine uncertainties, as the elevator predominantly produces a pitching moment and almost negligible lift force. The results for these kinds of uncertainties are shown in figure 12. It is obvious that in comparison to the baseline controller (figure 12.a)) the performance w.r.t. matched uncertainties can be improved by MRAC (figure 12.b)) and $\mathcal{L}_1$ (figure 12.c)) where the contour lines in figure 12.b) and c) refer to the baseline performance. Furthermore, the performance is assessed for uncertainties in the control effectiveness λ, which is equivalent to an uncertain gain in the input channel of the plant. The results are shown in figure 13.a)-c) and here it can be seen that over a large range the adaptive controllers improve the performance in comparison to the baseline controller. So in general it can be summarized that both controllers provide the possibility to improve the robust performance w.r.t. matched uncertainties and uncertainties in the input channel. Of course one could further reduce the bandwidth in the input channel in both approaches, but this would only lead to reduced performance.

Table 5 Parameter for load factor response boundaries

HQ Level 1	HQ Level 2	HQ Level 3
Overshoot < 0.10 80% Rise time < 4s 5% Settling time < 6s 1% Settling time < 10s	Overshoot < 0.20 80% Rise time < 6s 5% Settling time < 8s 1% Settling time < 12s	Overshoot < 0.30 80% Rise time < 8s 5% Settling time < 10s 1% Settling time < 14s

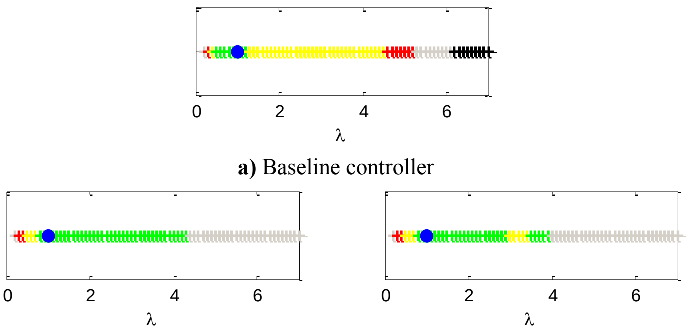


a) Baseline controller

b) MRAC controller

c) $\mathcal{L}_1$ controller

Fig. 12 Robust performance w.r.t. M_α and M_q



a) Baseline controller

b) MRAC controller

c) $\mathcal{L}_1$ controller

Fig. 13 Robust performance w.r.t. λ

5 Conclusion

As it was shown for the case where the control effectiveness is known, hedging and $\mathcal{L}_1$ adaptive control are exactly the same and the theory of $\mathcal{L}_1$ adaptive control can be used to provide a stability proof for the modified reference model. This also means that the same performance guarantees as provided by $\mathcal{L}_1$ adaptive control hold. However for the case were the control effectiveness is unknown the $\mathcal{L}_1$ approach differs from MRAC with hedging because it is driven by the stability and performance proof and therefore applies a filter where the bandwidth is adjusted by the estimated control effectiveness. Although analytic performance bounds might not be available for MRAC this does not mean that the approach provides worse performance. This could also be verified by the simulation example where both methods provide approximately the same robust performance.

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