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Optimal control gain for satellite detumbling using B-dot algorithm

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Abstract Theoretical derivation of the optimal control gain in the detumbling process using B-dot control law is presented. It is shown that the optimal gain is a function both of magnitude of magnetic field B and the rate of change of its direction. As both factors change along the orbit, the control gain applied should be variable.

Introduction

During last decade an increased interest in small satellite missions has been observed. These inexpensive, cost-effective spacecrafts tends to perform increasingly complex missions, which in turn require attitude stabilization. Volume and mass constraints limit the range of attitude control methods and mechanisms which can be successfully deployed. Magnetic actuation and acquisition is often the way to go in such applications - some deployment examples include NPSAT1[7], REIMEI [8] and Oersted satellite[12]. There is broad literature covering the area of magnetic actuation. Wiśniewski and Blanke proposed attitude controller derived using sliding mode approach in [6], Lovera and Astolfi present low-gain PD-like control law in [9], Zanchettin and Lovera derive H_∞ control law in [4], where they model process as linear periodic system. Some authors use magnetic actuation in pair with more conventional, momentum exchange devices. Authors of papers [10,11] combine it with momentum wheel (flywheel) which increases the system's stability. Despite many advantages, pure magnetic control introduces some

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problems. Due to small magnitude of Earth magnetic field control torques are severely restricted. What is more, coils are capable of generating torques lying in the plane perpendicular to local Earth magnetic field vector, what effectively makes the system underactuated. Small satellites often suffer from quick rotation gained in the process of deployment from the launcher rocket. This rotational movement prevents the attitude observer from synchronization with satellite dynamics using magnetometer readouts. Therefore simple and reliable control laws, which do not require attitude information, are often used to dissipate spacecraft's angular momentum and are commonly referenced as *detumbling mechanisms.*

A de facto standard for detumbling of very small satellites is called B-dot control, which uses only magnetorquers as actuators, and only magnetometer as sensor. The control law is build on time derivative of the measured magnetic field vector B, thus the name B-dot.

$$m = -K \cdot \dot{B} \qquad (1)$$

The magnetic dipole m generated with use of a magnetorquer (usually three mutually perpendicular coils) is proportional to the time derivative of B, performed in non-inertial satellite frame and gain K. This control law is described in numerous publications, where it is often proven [1] that the time derivative of kinetic energy of the satellite under B-dot control is given by (2), where ψ is vector of rotation rate of the satellite with respect to the frame defined by the magnetic field direction.

$$\dot{E}_k = -K \cdot \left\|\dot{B}\right\|^2 = -K \cdot \left\|\psi \times B\right\|^2 \qquad (2)$$

Due to the underactuated nature of system control gain tuning is not a straightforward task. Too much gain extends the detumbling time, therefore careful simulations are usually performed to find adequate gain value [2,3,4]. In the following chapter we would like to propose an analytical approach for determining the optimal control gain value.

Analytical solution

There are three sources of time derivative of measured field B: 1) the dominating one is rotation of the satellite with rate ω, 2) change of direction of B in the inertial frame as the satellite moves along the orbit, and 3) change of magnitude of B. The third effect can be neglected as the component parallel to B does not generate any torque. The second – change of direction of B, can not be ignored, as in fact thanks to this change the detumbling with magnetic control is at all possible. With constant B the parallel to B component of angular momentum could not be damped.

Let Ω_B be a vector representing rotation rate of B. It is perpendicular to B. Then time derivative of measured B can be expressed as (3).

$$^{S}\dot{B} = (\Omega_B - \omega) \times B \qquad (3)$$

Notation remark: $^{S}\dot{x}$ means that the time derivative of vector x is determined in the frame S (satellite) and next the vector of derivative is transformed to the frame specific for the equation. Shorter form $\dot{x}$ means that the derivative is determined in the same frame as of equation. If the frame S rotates with rate ω with respect to another frame, for example I (inertial), then holds:

$$^{I}\dot{x} = {}^{S}\dot{x} + \omega \times x \qquad (4)$$

where all components must of course be expressed in the same, thaw arbitrary frame.

Equation of dynamics of rigid body takes form (5), where ω is vector of rotation rate of the satellite with respect to the inertial frame. T is vector torque acting on the body, and I is inertia tensor matrix. This equation however is valid only in the body reference frame.

$$^{I}\dot{\omega} = I^{-1}(T - \omega \times I\omega) \qquad (5)$$

For our purposes a simplification is made – we assume spherical symmetry of mass distribution and then inertia tensor I becomes scalar. Authors realize that this assumption limits usage of analytical results. So far we can not find solution for the more general case.

Equation of dynamics takes more friendly form (6). It is important that this equation is valid in an arbitrary reference frame.

$$^{I}\dot{\omega} = I^{-1}T \qquad (6)$$

Torque T is result of B-dot control. Magnetic dipole m (1) in field B is source of torque (7).

$$T = m \times B \qquad (7)$$

Taking it altogether we get (8).

$$^{I}\dot{\omega} = -K \cdot I^{-1} \cdot B \times (\omega - \Omega_B) \times B \qquad (8)$$

We would like to solve this equation for $\omega(t)$. To make the task easier let's rewrite (8) in such reference frame in which both B and Ω_B are constant, let's call this frame M (magnetic). Let B be aligned with axis X and Ω_B with axis Z. Vector Ω_B is almost constant, so we can say that frame M rotates with rate Ω_B with respect to inertial frame, and the rule (4) can be used. New variable $\psi = \omega - \Omega_B$ is next used rather than ω. It is rotation rate vector of the satellite with respect to the frame M.

$$^{M}\dot{\psi} = -K \cdot I^{-1} \cdot B \times \psi \times B - \Omega_B \times \psi \qquad (9)$$

The index M at the derivative can be dropped, as the equation is considered in the same frame M. So, (9) is a linear differential equation for $\psi(t)$, which has exact analytical solution $\psi(t) = e^{At} \cdot \psi(0)$ where matrix

$A = K \cdot I^{-1} \cdot \left[B^{\times}\right]^{2} - \left[\Omega_B^{\times}\right]$. Symbol $\left[v^{\times}\right]$ denotes a skew symmetric matrix 3x3 build from elements of vector *v*.

$$[v^{\times}] \equiv \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

With the assumptions made so far the matrix *A* is equal to (10)

$$A = \begin{bmatrix} 0 & \Omega_B & 0 \\ -\Omega_B & -2\Omega_C & 0 \\ 0 & 0 & -2\Omega_C \end{bmatrix} \qquad (10)$$

$$\Omega_C \equiv \frac{1}{2} K \cdot I^{-1} \cdot B^2 \qquad (11)$$

Value Ω_C is an alternative to *K* measure of strength of control. It can be seen in matrix *A* that evolution of vector ψ in plane XY runs independently from the evolution of its component Z. This fact simplifies farther analysis, as there are two uncoupled solutions: one for ψ_{xy} and the other for ψ_z. Calculation of the matrix e^{At} can be performed with use of Laplace transform method, where one should calculate reverse transform: $e^{At} = L^{-1}\left[(sI - A)^{-1}\right]$. Solutions are presented in (12a,b).

$$\psi_z(t) = e^{-2\Omega_C t} \cdot \psi_z(0) \qquad (12a)$$

$$\psi_{xy}(t) = e^{-\Omega_C t} \cdot C(\Omega_B, \Omega_C, t) \cdot \psi_{xy}(0) \qquad (12b)$$

Component ψ_z quickly converges to zero with time constant $\tau_z = 1/(2\Omega_C)$. This means that vector ψ quickly falls onto the plane XY of the frame M, which is the plane in which the vector B rotates, and next the process is played in this plane. Character of changes of $\psi_{xy}(t)$, defined by the form of matrix $C_{[2x2]}$, depends on value of Ω_C with respect to Ω_B.

$$C = \cosh \Omega_0 t \cdot I_{2\times 2} + \frac{\sinh \Omega_0 t}{\Omega_0} \cdot \begin{bmatrix} \Omega_C & \Omega_B \\ -\Omega_B & -\Omega_C \end{bmatrix} \qquad (\Omega_C > \Omega_B)$$

$$C = \cos \Omega_0 t \cdot I_{2\times 2} + \frac{\sin \Omega_0 t}{\Omega_0} \cdot \begin{bmatrix} \Omega_C & \Omega_B \\ -\Omega_B & -\Omega_C \end{bmatrix} \qquad (\Omega_C < \Omega_B)$$

$$C = I_{2\times 2} + t \cdot \begin{bmatrix} \Omega_C & \Omega_B \\ -\Omega_B & -\Omega_C \end{bmatrix} \qquad (\Omega_C = \Omega_B)$$

where $\Omega_0 \equiv \sqrt{\left|\Omega_C^2 - \Omega_B^2\right|}$.

It is of course only a problem of mathematics that without using of complex numbers no single formula exists for the whole range of parameter Ω_C. Solutions are continuous with respect to this parameter, so the character of changes for Ω_C close to Ω_B, not necessarily exactly equal, can be deduced on base of the third form. The character of damping in this case is of type $t \cdot e^{-\Omega_C t}$, which is only a little slower than purely exponential.

It can be shown that the solution for ($\Omega_C > \Omega_B$) with time converges to (13).

$$\psi_{xy}(t) \cong \frac{e^{-(\Omega_C - \Omega_0)t}}{2\Omega_0} \begin{bmatrix} \Omega_0 + \Omega_C & \Omega_B \\ -\Omega_B & \Omega_0 - \Omega_C \end{bmatrix} \cdot \psi_{xy}(0) \quad (13)$$

Please note that the bigger Ω_C (bigger gain K) the smaller factor ($\Omega_C - \Omega_0$) in exponent, thus the slower decay of satellite rotation. Physical explanation is the following: the stronger gain K the smaller the angle between angular momentum and vector B. As only the perpendicular to B component of angular momentum is damped the small angle makes the damping non effective. In the extreme case of infinite K the two vectors are parallel and then no damping takes place.

Solution (13) has another interesting feature, which justifies calling it pseudo-stationary: the ratio ψ_y / ψ_x, which is equal to tangent of an angle between ψ and B, is constant and does not depend on the initial $\psi(0)$.

$$\tan\alpha = \frac{\psi_y}{\psi_x} = -\frac{\Omega_B}{\Omega_C + \Omega_0} = -\frac{\Omega_C - \Omega_0}{\Omega_B}$$

This means that the vector ψ follows the escaping vector B, keeping constant angular distance α. The same conclusion holds for the critical solution ($\Omega_C = \Omega_B$) in which case $\alpha = 45°$, and this is the largest possible angle for pseudo stationary state.

When control is weaker than the critical one ($\Omega_C < \Omega_B$) the behavior of $\psi_{xy}(t)$ looks a bit chaotic – it is a mixture of oscillations and exponential decay with time constant $\tau_{xy} = \Omega_C^{-1}$.

Optimization

Considerations set out in the previous section lead to believe that there is an optimal Ω_C assuming a criterion that evaluates convergence of $\psi(t)$. Character of convergence of $\psi(t)$ is close to exponential, so time constant of decay seems to be an adequate measure of speed of convergence. Please consider the equation

$\int_0^{\infty}\left(e^{-t/\tau}\right)^2 dt = \frac{\tau}{2}$ which gives a clue as how to calculate this time constant. Expression (14) can be understood as half of the time constant of decay of $|\psi(t)|$, which next can be subject of optimization.

$$J = \frac{1}{|\psi_0|^2}\int_0^{\infty}|\psi(t)|^2\,\mathrm{d}t \qquad (14)$$

The criterion J occurs to be dependent on a direction of the initial state vector ψ_0 in space XYZ, so before optimization some farther work must be done. The most common way to remove this drawback is to choose a maximum (15) or integral (16) over all initial conditions of unit norm [13].

$$J_1 = \max_{\|\psi_0\|=1}\int_0^{\infty}|\psi(t)|^2\,\mathrm{d}t \qquad (15)$$

$$J_2 = \int_{\|\psi_0\|=1}\left(\int_0^{\infty}|\psi(t)|^2\,\mathrm{d}t\right)\mathrm{d}\sigma \qquad (16)$$

The integral in (15) and (16) can easily be calculated taking into account the fact that

$$\int_0^{\infty}|\psi(t)|^2\,\mathrm{d}t = \int_0^{\infty}\psi^T(t)\psi(t)\,\mathrm{d}t = \psi_0^T\left(\int_0^{\infty}e^{A^Tt}e^{At}\,\mathrm{d}t\right)\psi_0 = \psi_0^T X\psi_0 \qquad (17)$$

where

$$X = \int_0^{\infty}e^{A^Tt}e^{At}\,\mathrm{d}t \qquad (18)$$

is a symmetric matrix which solves a Lyapunov equation [14]

$$A^T X + XA = -I \qquad (19)$$

Relationships (17) and (18) allow to present criteria (15) and (16) in a simple form [13, 15]

$$J_1 = \max_{\|\psi_0\|=1}\psi_0^T X\psi_0 = \|X\|_2 \qquad (20)$$

where $\|X\|_2$ is a spectral norm of matrix X, i.e. for a symmetric matrix the spectral norm is equal to the largest eigenvalue, and

$$J_2 = \int_{\|\psi_0\|=1}\psi_0^T X\psi_0\,\mathrm{d}\sigma = \mathrm{tr}\,X \qquad (21)$$

where $\mathrm{tr}\,X$ is a trace of matrix X.

We have conducted an analysis and optimization of the control system in two cases: (a) for reduced system described with a part of the matrix A (10), $A_r = \begin{bmatrix} 0 & \Omega_B \\ -\Omega_B & -2\Omega_C \end{bmatrix}$ having regard to the mutual independence of (ψ_x, ψ_y) and ψ_z, and (b) for the full matrix A. Case (a) has been studied due to consistency with the previous considerations.

The Table 1 shows results of optimization for both cases: derived functions describing the criteria $J_1(\Omega_C)$, $J_2(\Omega_C)$ and optimal values of Ω_C for each of the cases.

Table 1. Optimization criteria and results.

	Criterion	Optimal Ω_C
(a)	$J_1 = \frac{\Omega_C + \sqrt{\Omega_C^2 + \Omega_B^2}}{2\Omega_B^2} + \frac{1}{2\Omega_C}$	$\frac{\Omega_C}{\Omega_B} = \frac{(\sqrt{5}-1)\sqrt{2\sqrt{5}+2}}{4} \approx 0.79$
(a)	$J_2 = \frac{\Omega_C}{\Omega_B^2} + \frac{1}{\Omega_C}$	$\frac{\Omega_C}{\Omega_B} = 1$
(b)	$J_1 = \frac{\Omega_C + \sqrt{\Omega_C^2 + \Omega_B^2}}{2\Omega_B^2} + \frac{1}{2\Omega_C}$	$\frac{\Omega_C}{\Omega_B} = \frac{(\sqrt{5}-1)\sqrt{2\sqrt{5}+2}}{4} \approx 0.79$
(b)	$J_2 = \frac{\Omega_C}{\Omega_B^2} + \frac{5}{4\Omega_C}$	$\frac{\Omega_C}{\Omega_B} = \frac{\sqrt{5}}{2} \approx 1.18$

The resulting optimal values occur between 0.79-1.18 which is in line with the earlier findings. It is interesting that the criterion $J_1(\Omega_C)$ is described with the same function in the reduced and full model, and hence leads to identical results of optimization.

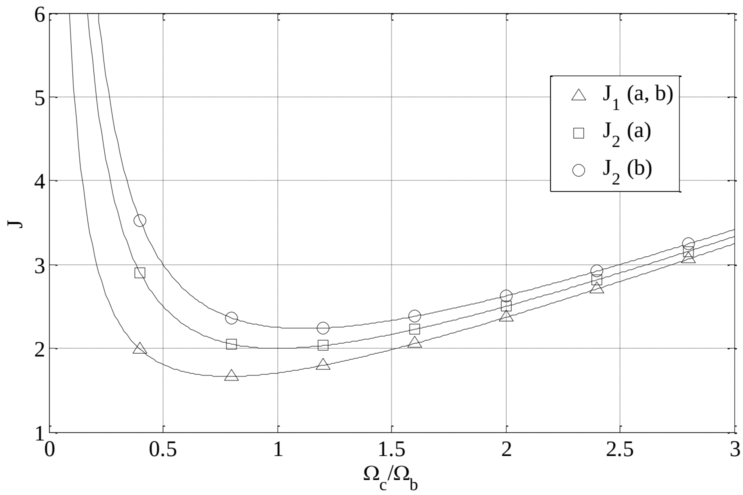

Fig. 1. Optimization criteria J_1 and J_2 in cases (a) and (b).

Not only speed of decay of the rotation rate of the satellite is interesting, but also energy of control spent in the detumbling process. This control energy is proportional to the integral of square of magnetic dipole generated over time (22).

$$E_C \propto \int_0^\infty m^2(t)\,\mathrm{d}t = \left(\frac{I}{B}\right)^2 \psi^2(0)\,\Omega_C \qquad (22)$$

Final choice of Ω_C should take into account both criteria: J and E_C. Solution can't be given here of course, because it depends on some weights being functions of the importance given by the user to effectiveness from one hand, and energy spent from the other hand.

Simulations

Results of simulations are given here to illustrate behavior of detumbling process for three values of the gain: optimal, three times weaker, and three times stronger. Simulated satellite has spherical symmetrical inertia with moment I = 0.008 kg m^2, and initially rotates with rate 10 deg/sec. Figures show one hour of simulation – magnitude of rotation rate and angle between vectors ω and B.

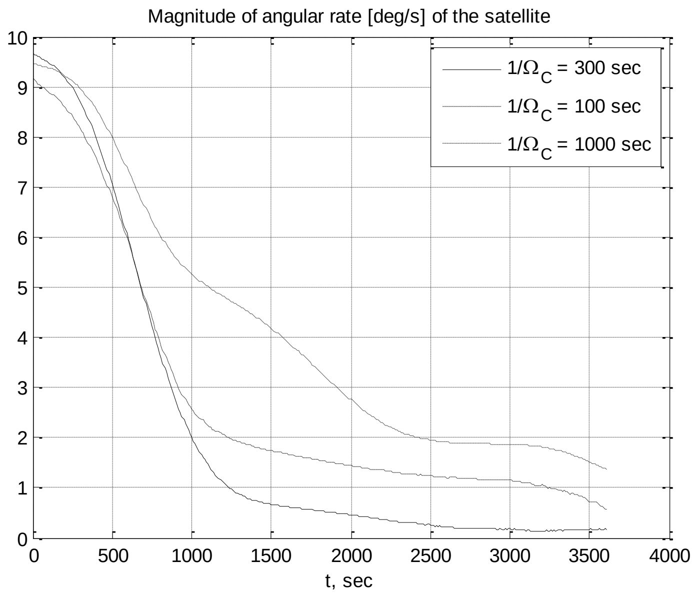

Fig. 2. One hour of simulation – magnitude of angular rate.

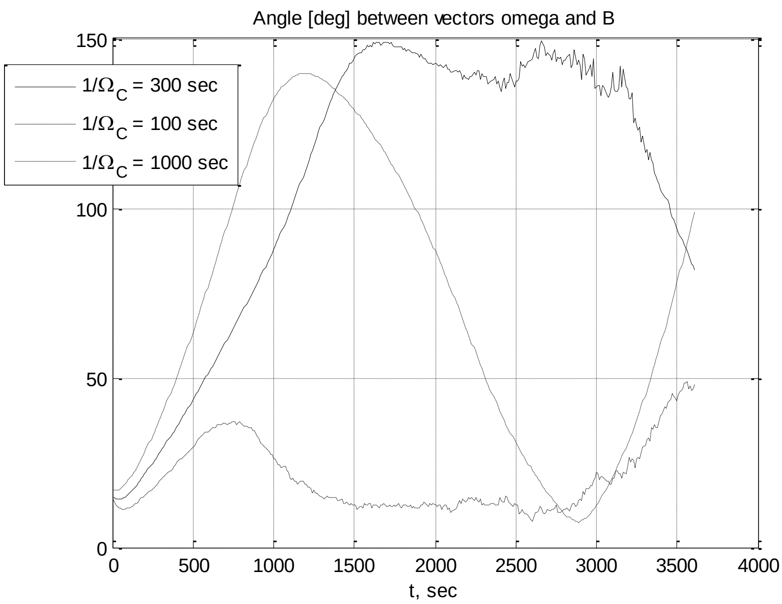

Fig. 3. One hour of simulation – angle between vectors ω and B.

Some of the features, pointed out before, are clearly visible:

1. The fastest decay of rotation rate takes place when the angle $\omega|B$ is close to 90 degrees. This state however is not stable.
2. For the stronger gain the process, after initial fast decay of rotation rate stabilize with small angle $\omega|B$ and then the damping is not effective.
3. For the optimal gain the angle $\omega|B$ stabilize at about 40 degrees (180–140=40) and this option is most effective at the end of the simulation period.
4. When the gain is weak, the vector ω can not follow escaping vector B, and it practically does not change direction. From one hand this is good, because the angle periodically passes the effective region close to 90 degrees, but with weak gain those lucky moments are not exploited enough.

Conclusions

From the point of view of effectiveness of detumbling the optimal control is achieved when Ω_C is close to Ω_B. From the other hand this gives the most chaotic evolution of the satellite rotation rate vector, as the mode wanders between pseudo-stationary and oscilatory. Please note that the true Ω_B is rarely known, and designer assumes an average value which is specific for a given orbit. For polar-like orbits this average is close to the orbital angular rate multiplied by two.

It is rather a matter of opinion if one accepts the chaos or seeks for a solution which gives more predictable behavior. Control, which is stronger than the critical one, ensures pseudo stationary behavior. From the other hand weaker control, below the critical one, brings another advantage – savings in control energy.

At the end we need a value of gain K. It is function (11) of Ω_B and B. Even for circular orbits the magnitude of magnetic field B can vary with the factor of three. Moreover it enters with square into the equation for K, so in real implementations it is desirable to make K dependent on the current (measured) B.

A number of simplifications was made to make the analytical solution easier or at all possible. Spherical symmetry of inertia was assumed. This of course limits usage of analytical results. So far we can not find solution for the more general case. The effectiveness of detumbling in case of non spherical inertia will be investigated with simulations in a future work.

We were considering the continuous time control model, while real implementations work in discrete time. The detumbling by its nature has to do with rather high angular rates, therefore during one time quantum the satellite rotates by a significant angle. This aspect must always be taken into account in a project – the sampling time must be small enough to ensure stability.

Real magnetorquers can generate only a limited magnetic dipole. This limit should be compared with the value $m_{\max} \approx \frac{2I\Omega_B \omega_{\max}}{B_{\min}}$, which estimates magnitude of the dipole required by the B-dot control. For example CubeSat 1U on a sun-synchronous orbit rotating 10 deg/sec would require $m_{\max}$ = 1 Am^2. This number exceeds real possibilities several times, so one should expect under-actuation, at least for higher angular rates.

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