Proceedings of the EuroGNC 2013, 2nd CEAS Specialist Conference on
Guidance, Navigation & Control, Delft University of Technology, Delft,
The Netherlands, April 10-12, 2013

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A frequency-limited $\mathcal{H}_2$ model approximation method with application to a medium-scale flexible aircraft

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Abstract In this paper, the problem of approximating a medium-scale MIMO LTI dynamical system over a bounded frequency range is addressed. A new method grounded on the SVD-Tangential model order reduction framework is proposed. Based on the frequency-limited gramians defined in [5], the contribution of this paper is to propose a *frequency-limited iterative SVD-Tangential interpolation algorithm* (FL-ISTIA) to achieve frequency-limited model approximation without involving weighting filters. The efficiency of the approach is addressed both on standard benchmark and on an industrial flexible aircraft model.

1 Introduction

1.1 Motivation

Computer-based modeling software are often used in order to accurately capture the mathematical model of physical systems or phenomena. They enable to handle complex systems with an enhanced accuracy. These models allow time and cost saving in the development process, but they often involve a large number of variables and thus require a lot of resources when analysed or simulated. Some modern *analysis*

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or *synthesis* tools may thereby become inefficient for such high dimensional models. A relevant approach to solve this issue is to *approximate the model* with a smaller one.

The reduction process can be subject to several constraints which depend on the purpose of the model. A commonly used constraint is the closeness between the reduced-order model input/output behaviour and the full-order one over all frequencies. Though it is a very interesting problem (see [9, 15, 17]), forcing models to be close over all frequencies may be too binding. Indeed (i) some frequencies are physically meaningless and can be viewed as uncertainties, (ii) in practice, actuators and sensors bandwidth are limited which make some frequencies irrelevant for control purpose and (iii) when vibration control has to be performed, some frequencies are more specifically of interest. Therefore considering the problem of reducing the full-order model such that a good approximation is found *over a bounded frequency range* can be more appropriate and appealing for engineers. This is the problem treated in this paper.

1.2 Projection-based problem statement

The reduction problem which consists in approximating a large-scale model by projection is recalled in Problem 1.

Problem 1. Given a continuous, stable and strictly proper MIMO LTI dynamical model Σ defined as

$$\Sigma := \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_u}$ and $C \in \mathbb{R}^{n_y \times n}$. The projection-based model order reduction problem consists in finding $V, W \in \mathbb{R}^{n \times r}$ with $W^T V = I_r$ such that the reduced-order model $\hat{\Sigma}$ of order $r \ll n$ defined as

$$\hat{\Sigma} := \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) = \hat{C}\hat{x}(t) \end{cases} \tag{2}$$

where $\hat{A} = W^T AV$, $\hat{B} = W^T B$ and $\hat{C} = CV$, accurately reproduces the behaviour of the full-order system Σ over the whole frequency domain.

The accuracy can be evaluated through the $\mathcal{H}_2$-norm of the error system $\Sigma - \hat{\Sigma}$. This measure, called mismatch error, is a good indicator of the global error between the models and is commonly used in many research papers [9, 15, 17].

1.3 Frequency-limited model approximation problem

In this paper, a similar formulation is addressed for the frequency-limited case (see Problem 2).

Problem 2. Given a continuous, stable and strictly proper MIMO LTI dynamical model Σ as in (1), the projection-based frequency limited approximation problem consists in finding projectors $V, W \in \mathbb{R}^{n \times r}$ in order to construct the reduced-order model $\hat{\Sigma}$ as in (2) such that $\hat{\Sigma}$ well approximates Σ over a given bounded frequency range.

In this paper, this problem will be tackled for the frequency range $[0, \omega]$ because low frequencies are particularly of interest for the intended applications. The accuracy of the approximation over $[0, \omega]$ will be evaluated through the frequency-limited $\mathscr{H}_2$-norm proposed in [11] and recalled later in Definition 2.

1.4 Paper structure

The paper is divided as follow. In Section 2 some preliminary results on the standard $\mathscr{H}_2$ model approximation are recalled. Then in Section 3, specific tools for frequency-limited model approximation are presented. In Section 4, the proposed frequency-limited approximation method is introduced. It is applied and compared on standard benchmark models and on an industrial flexible aircraft model in Section 5. Section 6 concludes this article.

2 Preliminary results on $\mathscr{H}_2$ model approximation

2.1 $\mathscr{H}_2$-Optimal model approximation

The model approximation problem formulated previously in Problem 1 can be viewed as the minimization of the following entity

$$\mathscr{J}_{\mathscr{H}_2}\left(\hat{A}, \hat{B}, \hat{C}\right) = \|\Sigma - \hat{\Sigma}\|^2_{\mathscr{H}_2} \tag{3}$$

which represents the mismatch error between the full-order and the reduced-order models in terms of $\mathscr{H}_2$-norm, i.e. over the whole frequency range (see Definition 1).

Definition 1 *The $\mathscr{H}_2$-norm of a stable and strictly proper system Σ whose transfer function is $H(s) = C(sI_n - A)^{-1}B$, is given by*

$$\begin{aligned}
\|H\|_{\mathscr{H}_2}^2 &= \frac{1}{2\pi j} \int_{-\infty}^{\infty} \mathbf{trace}\left(H(j\omega)H(-j\omega)^T\right) ds \\
&= \mathbf{trace}\left(B^T \mathscr{Q} B\right) \\
&= \mathbf{trace}\left(C \mathscr{P} C^T\right)
\end{aligned} \tag{4}$$

where $\mathscr{P}$ and $\mathscr{Q}$ are the controllability and the observability gramians given in the frequency domain by the following integrals:

$$\mathscr{P} = \frac{1}{2\pi} \int_{-\infty}^{\infty} T(\nu) B B^T T^*(\nu) d\nu \tag{5a}$$

$$\mathscr{Q} = \frac{1}{2\pi} \int_{-\infty}^{\infty} T^*(\nu) C^T C T(\nu) d\nu \tag{5b}$$

with $T(\nu) = (j\nu I - A)^{-1}$.

Minimizing $\mathscr{J}_{\mathscr{H}_2}$ is a non-convex problem, thus finding a global minimizer is a complex task. Finding a local minimizer is a more tractable problem. The most commonly used approach consists in derivating the first-order necessary optimality conditions which have been first addressed by Wilson [17]. Based on the interpolatory framework of Grimme [6], the *Iterative Rational Krylov Algorithm* (IRKA) proposed in [9] enables to fill these conditions and leads to a local minimizer. However due to numerical issues, it is rather dedicated to SISO systems. For MIMO systems, the tangential interpolatory framework [4] seems to be more appropriate. Equivalent first-order optimality conditions have also been derived for this case [15] and are recalled in Theorem 1.

Theorem 1 *If $\nabla_{\hat{A}} \mathscr{J}_{\mathscr{H}_2} = 0, \nabla_{\hat{B}} \mathscr{J}_{\mathscr{H}_2} = 0$ and $\nabla_{\hat{C}} \mathscr{J}_{\mathscr{H}_2} = 0$, which are the gradients of $\mathscr{J}_{\mathscr{H}_2}$ with respect to $\hat{A}$, $\hat{B}$ and $\hat{C}$ respectively, then the following tangential interpolation conditions are satisfied for $i = 1, \dots, r$:*

$$\begin{aligned}
[H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)]\hat{b}_i &= 0 \\
\hat{c}_i^*[H(-\hat{\lambda}_i) - \hat{H}(-\hat{\lambda}_i)] &= 0 \\
\hat{c}_i^* \frac{d}{ds}[H(s) - \hat{H}(s)]|_{s=-\hat{\lambda}_i} \hat{b}_i &= 0
\end{aligned} \tag{6}$$

where the $\hat{\lambda}_i$ are the eigenvalues of $\hat{A}$, $\{\hat{b}_1, \dots, \hat{b}_r\} = \hat{B}^T R$ and $\{\hat{c}_1, \dots, \hat{c}_r\} = \hat{C} L$ (where L and R are the left and right eigenvectors associated to $\hat{\lambda}_i$).

Theorem 1 expresses the necessary conditions to find a local minimum of $\mathscr{J}_{\mathscr{H}_2}$. Hence the optimal model approximation problem consists in finding $\left\{\hat{\lambda}_i, \hat{c}_i, \hat{b}_i\right\}$ such that (6) is satisfied. Theorem 2 then makes the link with Problem 1 and shows how the projectors V and W are constructed to fulfil these conditions.

Theorem 2 *Let $V \in \mathbb{C}^{n \times r}$ and $W \in \mathbb{C}^{n \times r}$ be full rank matrices such that $W^T V = I_r$. Let $\sigma_i \in \mathbb{C}^r$, $\hat{b}_i \in \mathbb{C}^{n_u}$ and $\hat{c}_i \in \mathbb{C}^{n_y}$ (for $i = 1, \dots, r$) be given sets of interpolation points and left and right tangential directions, respectively. Assume that points σ_i are selected such that $\sigma_i I_n - A$ are invertible. If, for $i = 1, \dots, r$,*

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$$
\begin{aligned}
(\sigma_i I_n - A)^{-1} B \hat{b}_i &\in \boldsymbol{span}(V) \\
and \ (\sigma_i I_n - A^T)^{-1} C^T \hat{c}_i^* &\in \boldsymbol{span}(W),
\end{aligned}
\tag{7}
$$

then, the reduced-order system $\hat{H}(s)$ satisfies the tangential interpolation conditions given in Theorem 1.

The *Iterative Tangential Interpolation Algorithm* (ITIA) suggested in [15] is a very efficient way to achieve Theorem 2. The IRKA and ITIA are numerically efficient and lead to local minimizers of $\mathscr{J}_{\mathscr{H}_2}$. Nevertheless they theoretically do not preserve stability of the full-order model[1] and can lead to poor approximant when applied to ill-conditioned models. Moreover for approximating medium-scale dynamical systems, numerical efficiency is less crucial than it can be in (very)large-scale cases. That is why it may be more adequate to use a method which is heavier than IRKA or ITIA from a computational point of view but which offers more guarantees and more robustness to parameters selection. Such a method has first been proposed by Gugercin in [7] and is called *Iterative SVD-Rational Krylov Algorithm* (ISRKA). It requires to compute only one gramian and it is directly applicable to SISO, MISO and SIMO systems. A similar algorithm for MIMO systems, called *Iterative SVD Tangential Interpolation Algorithm* (ISTIA) has been proposed in [13]. It is the basis of this work and for sake of completeness, its main properties are recalled there after.

2.2 ISTIA

This algorithm consists in using one single gramian to construct one of the two projectors involved in the approximation by projection. Indeed one projector is designed by solving one single Lyapunov equation while the second one is iteratively constructed to achieve one sided tangential interpolation and thus fulfil a subset of the optimality conditions presented in Theorem 1. For instance, V and W can be constructed such that

$$
\boldsymbol{span}(V) = \left[(\sigma_1 I_n - A)^{-1} B \hat{b}_1, \ldots, (\sigma_r I_n - A)^{-1} B \hat{b}_r \right]
\tag{8}
$$

where σ_i are the shift points and $\hat{b}_i$ corresponding right tangential directions, and

$$
W = \mathscr{Q} V (V^T \mathscr{Q} V)^{-1}
\tag{9}
$$

where $\mathscr{Q}$ is the observability gramian. See [7] and [13] for more details on the selection of interpolation points and for the complete version of the algorithm.

This method is numerically more expensive than the IRKA but it offers also more guarantees. Indeed, if the full-order model is stable, then the reduced-order

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[1] Yet in practice it is often the case. Moreover algorithmic procedures such as restarting can be used to avoid instability.

one will be stable as well. The proof can be found in [7]. It consists in considering that $\mathscr{Q} = I_n$. Hence, $W = V$ and $V^T V = I_r$. The Lyapunov equation becomes,

$$A^T + A + C^T C = 0 \tag{10}$$

By left and right multiplying (10) by V^T and V, it comes

$$\hat{A}^T + \hat{A} + \hat{C}^T \hat{C} = 0 \tag{11}$$

which indicates, by inertia results [12], that $\hat{A}$ is stable. For the same reasons as in [7], $\hat{A}$ is even asymptotically stable.

3 Preliminary results on frequency-limited model approximation

So far, only the $\mathscr{H}_2$ optimal model approximation has been considered but a lot of studies concern the model approximation over a bounded frequency range. Useful tools related to this issue are presented in this section.

3.1 The frequency-weighted approach

The most common approach to tackle the issue of reducing a model over a bounded frequency interval consists in considering input and/or output filters $W_i(s)$ and $W_o(s)$ so that the reduction is achieved on the filtered full-order system $\tilde{H}(s)$ given by

$$\tilde{H}(s) = W_o(s)H(s)W_i(s) \tag{12}$$

where $H(s) = C(sI_n - A)^{-1}B$. The weighted model reduction problem has often been tackled by weighted versions of the balanced truncation, see for instance [8] and references therein for an overview of these methods. More recently, this problem has been tackled from an interpolatory point of view, see [3] and [2].

Despite interesting results, the use of weights is very limiting since their choose is a time consuming and challenging task for engineers. For instance to achieve frequency-weighted balanced truncation, weights have to be stable and minimum phase filters. To alleviate this practical difficulty, a weight-free approach is preferred in this paper.

3.2 Frequency-limited gramians and balanced truncation

In [5], the authors proposed to narrow the frequency range of the integrals in (5a) and (5b) in order to get gramians in frequency interval $[0, \omega]$:

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$$\mathcal{P}_\omega = \frac{1}{2\pi} \int_{-\omega}^{\omega} T(v) BB^T T^*(v) dv \tag{13a}$$

$$\mathcal{Q}_\omega = \frac{1}{2\pi} \int_{-\omega}^{\omega} T^*(v) C^T C T(v) dv \tag{13b}$$

with $T(v) = (jvI_n - A)^{-1}$.

These gramians are solutions of the two following Lyapunov equations:

$$A\mathcal{P}_\omega + \mathcal{P}_\omega A^T + W_c(\omega) = 0 \tag{14a}$$
$$A^T \mathcal{Q}_\omega + \mathcal{Q}_\omega A + W_o(\omega) = 0 \tag{14b}$$

where

$$W_c(\omega) = S(\omega) BB^T + BB^T S^*(\omega) \tag{15a}$$
$$W_o(\omega) = S^*(\omega) C^T C + C^T C S(\omega) \tag{15b}$$

and

$$S(\omega) = \frac{1}{2\pi} \int_{-\omega}^{\omega} T(v) dv$$
$$= \frac{j}{2\pi} \mathbf{log} \left((A + j\omega I_n)(A - j\omega I_n)^{-1} \right) \tag{16}$$

A direct application of the frequency-limited gramians in model order reduction is the frequency-limited balanced truncation presented in [5]. It consists firstly in balancing $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$, that is to say to find a basis so that both gramians are equals and diagonals:

$$\mathcal{P}_\omega = \mathcal{Q}_\omega = \mathbf{diag} \left(\sigma_{n_1} I_{n_1}, \ldots, \sigma_{n_q} I_{n_q} \right) \tag{17}$$

where σ_i is a singular value with multiplicity n_i. Then the model is classically truncated to obtain the reduced-order model.

Since $W_c(\omega)$ and $W_o(\omega)$ are not positive semi-definite, the frequency-limited gramians $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$ are not guaranteed to be positive semi-definite (see [12]). Hence the reduced-order model obtained this way might be unstable. To preserve stability, [8] has proposed a modification to this method but it drastically impacts the quality of the reduced-order model.

As it is mentioned in [8], using frequency-limited gramians for balanced truncation can be viewed as a frequency-weighted balanced truncation method with *perfect filters*.

3.3 $\mathcal{H}_{2,\omega}$-norm: frequency-limited $\mathcal{H}_2$-norm

The $\mathcal{H}_2$-norm is a convenient metric for measuring the quality of an approximant *over the whole frequency range*, however it is less relevant if the approximant has to be good only over a finite frequency interval $[0, \omega]$. In this case, another metric has to be considered.

A frequency-bounded $\mathcal{H}_2$-norm has been addressed in [1] and recalled more recently in [11] as a restriction of the $\mathcal{H}_2$-norm over the frequency range $[0, \omega]$. Its state-space representation directly comes from the definition of the frequency-limited gramians given in Definition 2.

Definition 2 *Given a stable and strictly proper MIMO linear dynamical system Σ with $H(s) = C(sI_n - A)^{-1}B$, the $\mathcal{H}_{2,\omega}$-norm is defined as follow*

$$\begin{aligned}
\|H\|_{\mathcal{H}_{2,\omega}}^2 &= \frac{1}{2\pi} \int_{-\omega}^{\omega} \mathbf{trace}\left(H(j\nu)H(-j\nu)^T\right) d\nu \\
&= \mathbf{trace}\left(C\mathcal{P}_\omega C^T\right) \\
&= \mathbf{trace}\left(B^T \mathcal{Q}_\omega B\right)
\end{aligned} \tag{18}$$

where $\mathcal{P}_\omega$ and $\mathcal{Q}_\omega$ are the frequency-limited gramians defined by (13a) *and* (13b)*.*

Remark 1 *Frequency-limited gramians can also be expressed over the frequency interval $\Omega = [\omega_1, \omega_2]$. Indeed, $\mathcal{P}_\Omega = \mathcal{P}_{\omega_2} - \mathcal{P}_{\omega_1}$ and $\mathcal{Q}_\Omega = \mathcal{Q}_{\omega_2} - \mathcal{Q}_{\omega_1}$. Hence a restriction of the $\mathcal{H}_2$-norm over the interval Ω can easily be expressed in a similar way.*

Property 1 *If $H(s)$ is a stable and strictly proper linear dynamical system, then its frequency-bounded $\mathcal{H}_2$-norm tends towards its $\mathcal{H}_2$-norm when the frequency bound tends towards infinity,*

$$\|H\|_{\mathcal{H}_{2,\omega}} \xrightarrow[\omega \to \infty]{} \|H\|_{\mathcal{H}_2} \tag{19}$$

Proof. Applying the residue theorem in (18) leads to the result. Another proof can be found in [16]. $\qquad\square$

To illustrate how the $\mathcal{H}_{2,\omega}$-norm behaves as a function of ω, the Los-Angeles hospital model is used (see [10]). It has 48 states, 1 input and 1 output. Its frequency response and its $\mathcal{H}_{2,\omega}$-norm are computed for several values of ω on Figure 1. It enables to illustrate Property 1 and the fact that the $\mathcal{H}_{2,\omega}$-norm evolves by steps. When ω crosses the abscissa of a peak of the frequency response, the $\mathcal{H}_{2,\omega}$-norm of $H(s)$ crosses a step. This can be viewed as the contribution of the gain of the peak in the global input/output energy represented by the $\mathcal{H}_2$-norm.

4 Frequency-limited Iterative SVD-Tangential Interpolation Algorithm

The proposed algorithm, namely the *Frequency-Limited Iterative SVD-Tangential Interpolation Algorithm*, or FL-ISTIA (see Algorithm 1) is very similar to the IS-TIA. Indeed, one projector is built through tangential interpolation (step 1 and 9)

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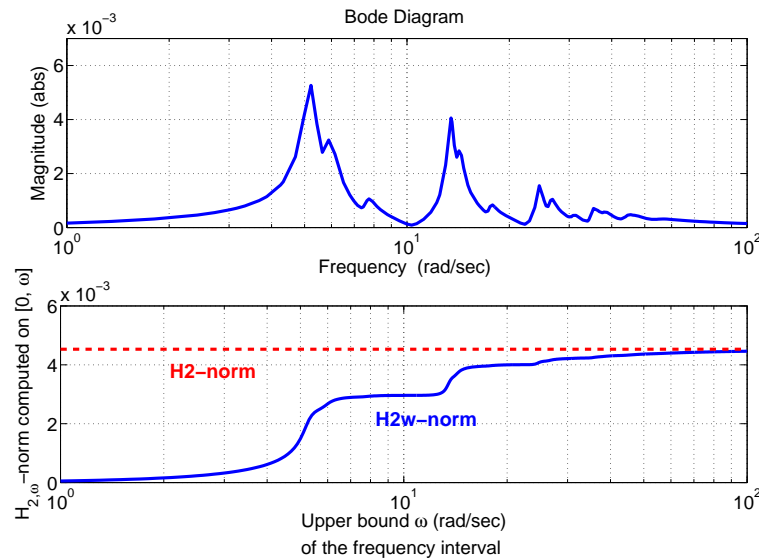
Fig. 1 Evolution of the $\mathscr{H}_{2,\omega}$-norm against the frequency (Los Angeles hospital model)

whereas the other is obtained through the computation of a gramian. The main difference lies in the fact that the gramian used to build the second projector is a frequency-limited gramian.

For numerical purpose, the right projector W is obtained by enforcing *orthogonality*, as in step 3. Then, from step 4 to 11, the construction of projectors is repeated by using as new interpolation points the *mirror images* of the eigenvalues of the reduced-order model, and, as new interpolation directions, the *right eigenvectors* associated with these eigenvalues (steps 6-8). The process is repeated until the interpolation points variation is smaller than a user defined tolerance ε.

Some remarks about this algorithm can be addressed :

- Unlike the ISTIA, the stability of the reduced-order model cannot be guaranteed. Indeed, since $\mathscr{Q}_\omega$ (step 2) is not guaranteed to be positive semi-definite, the reasoning done previously with the ISTIA in Section 2.2 is no longer valid. Yet in practice, instability has not been observed and numerical procedures such as *restarting* can be used to alleviate this drawback.
- The initial shift points are selected so that their modulus is less than the frequency bound ω, i.e. $|\sigma_i^{(0)}| \leq \omega$, $i = 1, \ldots, r$. It is done to favour the interpolation of the full-order model under this bound. A similar constraint could be imposed on the following interpolation points (step 8) but the selection of tangential directions would then become an issue.
- The frequency-limited controllability gramian $\mathscr{P}_\omega$ can identically be used instead of the observability one, $\mathscr{Q}_\omega$. In this case, the tangential subspace to be con-

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Algorithm 1 Frequency-Limited Iterative SVD-Tangential Interpolation Algorithm (FL-ISTIA)

Require: $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n_u}, C \in \mathbb{R}^{n_y \times n}, R \in \mathbb{R}^{+*}, \{\sigma_1^{(0)}, \ldots, \sigma_r^{(0)}\} \in \mathbb{C}^{n \times r}$ with $|\sigma_i^{(0)}| \leq R, i = 1, \ldots, r, \{\hat{b}_1, \ldots, \hat{b}_r\} \in \mathbb{C}^{n \times r}, \varepsilon > 0$

1: Construct,
$$\mathbf{span}(V) = \left[(\sigma_1^{(0)} I_n - A)^{-1} B \hat{b}_1, \ldots, (\sigma_r^{(0)} I_n - A)^{-1} B \hat{b}_r \right]$$

2: Solve $\mathcal{Q}_\omega A + A^T \mathcal{Q}_\omega + W_o(\omega) = 0$ in $\mathcal{Q}_\omega$
3: Compute $W = \mathcal{Q}_\omega V (V^T \mathcal{Q}_\omega V)^{-1}$
4: **while** $|\sigma^{(i)} - \sigma^{(i-1)}| > \varepsilon$ **do**
5:      $i \leftarrow i+1, \hat{A} = W^T A V, \hat{B} = W^T B$
6:      Compute $\hat{A} X = \mathbf{diag}(\lambda(\hat{A})) X$
7:      Compute $[\hat{b}_1, \ldots, \hat{b}_r] = \hat{B}^T X^{-T}$
8:      Set $\sigma^{(i)} = -\lambda(\hat{A})$
9:      Construct,
$$\mathbf{span}(V) = \left[(\sigma_1^{(i)} I_n - A)^{-1} B \hat{b}_1, \ldots, (\sigma_r^{(i)} I_n - A)^{-1} B \hat{b}_r \right]$$

10:      Compute $W = \mathcal{Q}_\omega V (V^T \mathcal{Q}_\omega V)^{-1}$
11: **end while**
12: Construct $\hat{\Sigma} : (W^T A V, W^T B, C V)$
Ensure: $V, W \in \mathbb{R}^{n \times r}, W^T V = I_r$ and $\mathbf{Re}\left(\lambda(\hat{A}) \right) < 0$

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structed is $\mathbf{span}(W) = \left[(\sigma_1^{(i)} I_n - A^T)^{-1} C^T \hat{c}_1, \ldots, (\sigma_r^{(i)} I_n - A^T)^{-1} C^T \hat{c}_r \right]$ (where $\{\hat{c}_1, \ldots, \hat{c}_r\} = \hat{C} X$).

- As in all Krylov-like procedures, to obtain real valued projection V and W matrices and increase computation speed, the starting shift grid should be either real or complex conjugate. Indeed, one can use the fact that, if, $v_2 = v_1^*$, then $\mathbf{span}[v_1, v_2] = \mathbf{span}[\mathbf{Re}(v_1), \mathbf{Im}(v_1)]$.
- Since this procedure requires to solve a n-th order Lyapunov equation, it is limited to medium-scale dynamical systems. It could be extended to larger systems with the use of low rank approximations of the gramian.
- The FL-ISTIA is equivalent to the ISTIA as ω increases. Indeed, as ω increases, the realisation given by the FL-ISTIA tends (element-wise) towards the one given by the ISTIA. This comes from the fact that frequency-limited gramians tends (element-wise) towards infinite gramians as ω tends towards infinity.

5 Applications

In this section, the Iterative SVD-Tangential Interpolation Algorithm (ISTIA), the frequency-limited balanced truncation (FL-BT) and the Frequency-Limited Iterative SVD-Tangential Interpolation Algorithm (FL-ISTIA) are compared through two standard benchmarks and one industrial flexible aircraft model.

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The quality of the approximation over $[0, \omega]$ is evaluated through the $\mathscr{H}_{2,\omega}$-norm (see Definition 2) of the relative error ε_ω, i.e.:

$$\varepsilon_\omega = \frac{\|\Sigma - \hat{\Sigma}\|_{\mathscr{H}_{2,\omega}}}{\|\Sigma\|_{\mathscr{H}_{2,\omega}}} \tag{20}$$

### 5.1 Standard benchmark models case

As a first application of the Iterative Frequency-Limited SVD Tangential Interpolation Algorithm (FL-ISTIA), the clamped beam model is used. It is a standard benchmark model [10] with 348 states, 1 input and 1 output.

The clamped beam model is reduced to order $r = 12$ using the three reduction methods. The upper bound ω of the frequency interval of reduction $[0, \omega]$ is gradually increased from 2rad/s to 20rad/s. Results are represented in Figure 2.
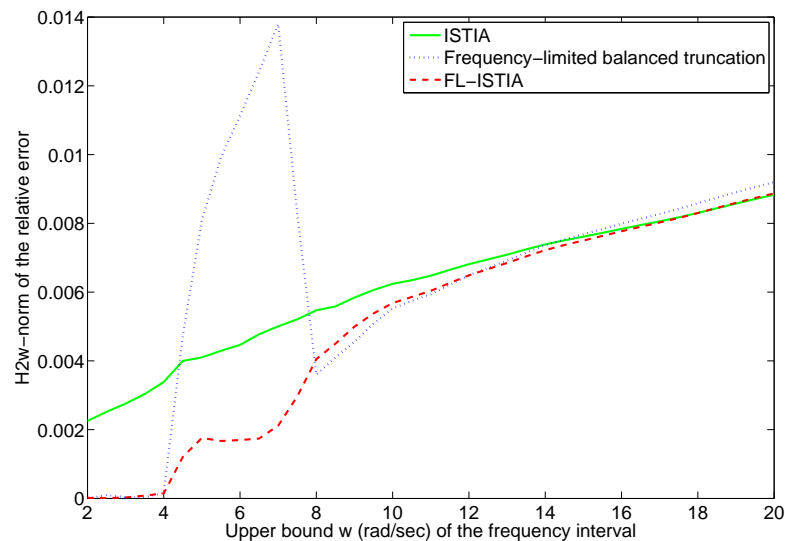
Fig. 2 $\mathscr{H}_{2,\omega}$-norm of the relative error against the upper frequency bound ω (clamped beam model, $r = 12$)

On this example, the FL-BT and the FL-ISTIA are quite similar excepted from 4rad/s to 8rad/s where the first method fails to correctly reduce the model. This may come from numerical issues related to the computation of frequency-limited gramians or the balancing of the system. This suggests that the FL-ISTIA is numerically more reliable.
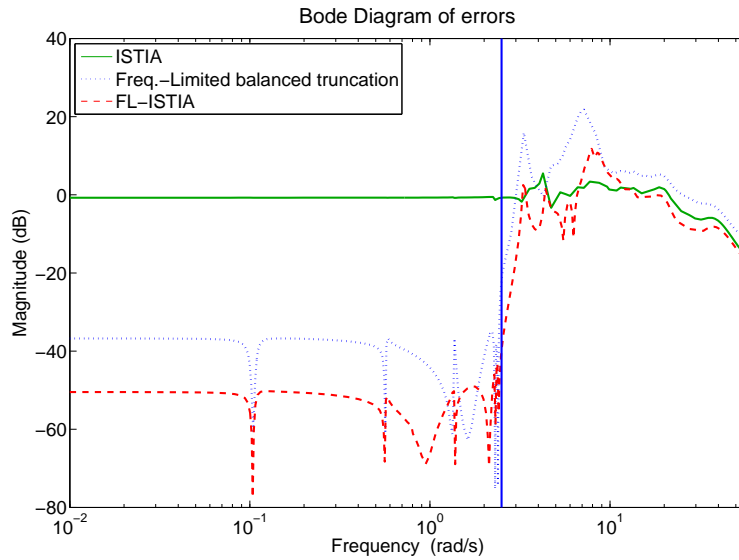
Fig. 3 Frequency responses of the error system for $\omega = 2,5\text{rad/s}$ (clamped beam model, $r = 12$)

Figure 2 also clearly illustrates the fact that the FL-ISTIA and the ISTIA become equivalent as ω increases since they lead to the same reduced-order model. The frequency-limited aspect of the approximation methods considered here is well illustrated by Figure 3 where the frequency responses of the error systems are plotted. The upper bound of the frequency interval used in this case was $\omega = 2,5\text{rad/s}$. It appears that the error is very low from 0 to $\omega = 2,5\text{rad/s}$ and it rises after this bound.

Similar results can be observed when the procedure is applied to the Los Angeles hostpital model (see Figure 4).

5.2 Flexible aircraft model case

The second application is done in a similar way on a flexible aircraft model which comes from the industry. It has 289 states, 4 outputs and 3 inputs.

The flexible behaviour of an aircraft leads to a model with poorly damped modes, i.e. eigenvalues close to the imaginary axis and its rigid behaviour leads to eigenvalues which are real or almost real and very close to 0. All this make the model very ill-conditioned and thus hard to reduce with classical approaches [13].

The full-order model is reduced to order $r = 12$ by the three reduction methods. The upper bound R of the frequency interval goes from 1rad/s to 40rad/s and the $\mathcal{H}_{2,\omega}$-norm of the relative error is plotted with respect to ω on Figure 5.

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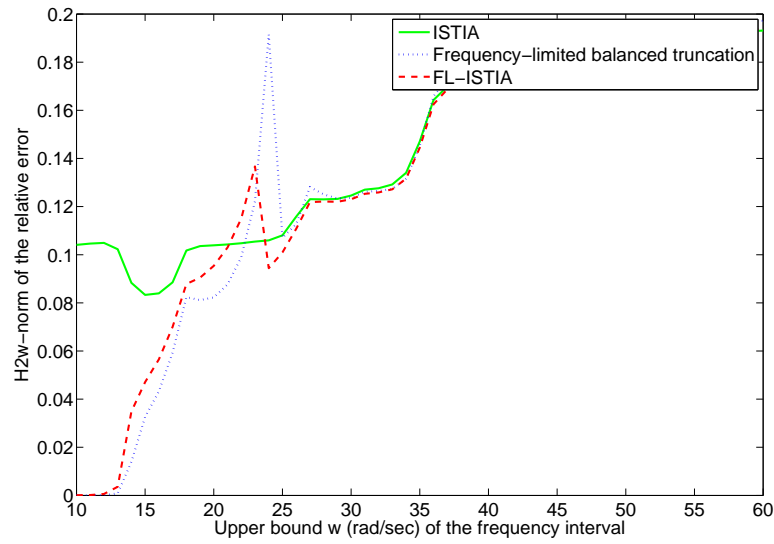
Fig. 4 $\mathcal{H}_{2,\omega}$-norm of the relative error against the upper frequency bound ω (Los Angeles hospital model, $r = 12$)
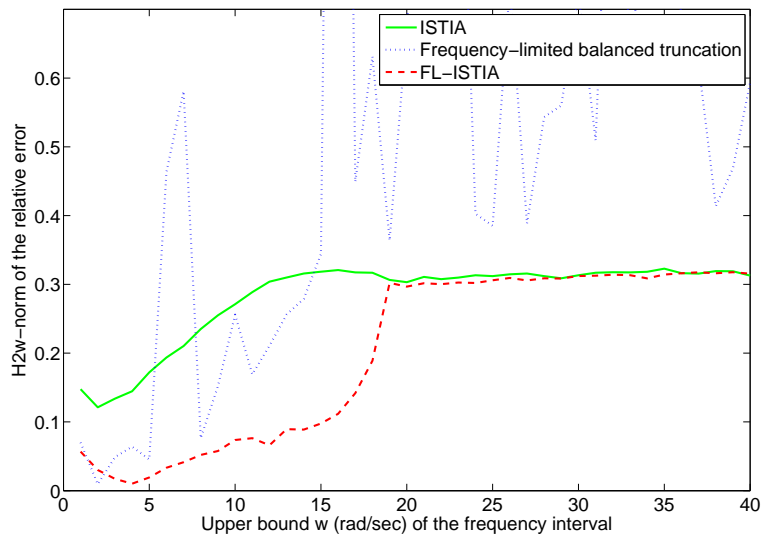
Fig. 5 $\mathcal{H}_{2,\omega}$-norm of the relative error with respect to the upper frequency bound ω (aircraft model, $r = 12$)
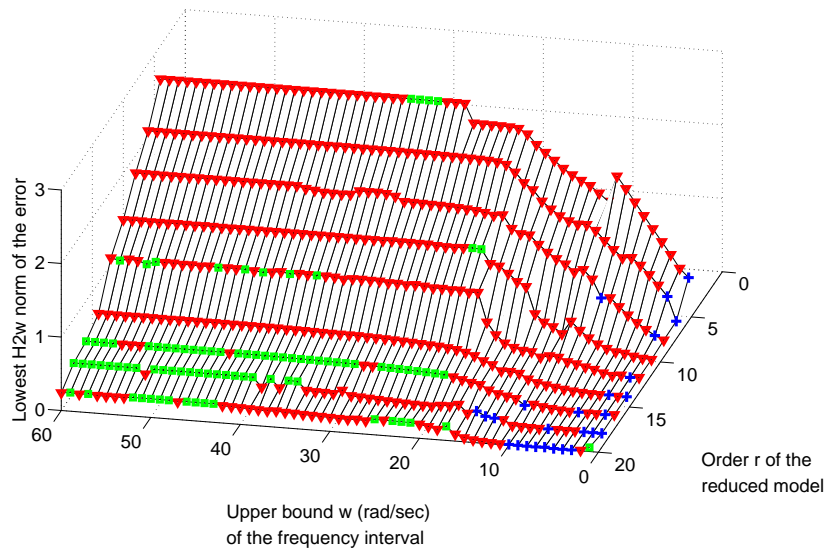
Fig. 6 $\mathcal{H}_{2,\omega}$-norm of the error with respect to the upper frequency bound ω and the order r of the reduced model (aircraft model)

Here the FL-BT leads to poor reduced-order models. The fact that the model is ill-conditioned increases the numerical issues arising in the computation of the frequency-limited gramians and in the balancing of the system. One example of those numerical issues is illustrated by the 2-norm of the Lyapunov equation

$$r_\omega = \|A^T \mathscr{Q}_\omega + \mathscr{Q}_\omega A + W_o(\omega)\|_2$$

which should be almost equal to zero. Yet, for $\omega = 14\text{rad/s}$, $r_\omega > 10^3$.

This error on the frequency-limited gramians directly impacts the FL-BT whereas it has little consequences on the FL-ISTIA. Indeed, until 18rad/s, FL-ISTIA leads to a better reduced-order model than the ISTIA and for larger ω they become equivalent. This equivalence comes from the fact that most of the spectral information is gathered in $0-20\text{rad/s}$.

When using the FL-BT and the FL-ISTIA, two parameters can be adjusted for the approximation: the upper bound ω of the frequency interval and the order r of the reduced model. Figure 6 represents the best approximation in terms of $\mathcal{H}_{2,\omega}$-norm among those provided by the ISTIA (green squares), the FL-BT (blue crosses) and the FL-ISTIA (red triangles) for several frequencies going from 1rad/s to 60rad/s and approximation orders going from $r = 4$ to $r = 20$. It plots the lowest $\mathcal{H}_{2,\omega}$ error, that is to say that when a method is better than the other, then its $\mathcal{H}_{2,\omega}$ error is plotted.

For this model it is clear that the FL-ISTIA mostly leads to a better approximation than the FL-BT independently of the frequency and order. Indeed, red triangles are

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predominant excepted for small frequencies ω and large order r where the FL-BT is the best method. This can be explained by the fact that a large number of interpolations cannot be achieved over a tight frequency range if there are not enough different behaviours to catch.

6 Conclusion

In this paper, a new application of the frequency-limited gramians proposed in [5] has been presented. Indeed they have been used in the *Iterative SVD-Tangential Interpolation Algorithm* (ISTIA, [13]) instead of infinite gramians which leads to an extended frequency-limited version of this algorithm called FL-ISTIA.

The relevance of the FL-ISTIA has been illustrated through two standards benchmark models and one flexible aircraft model. These tests have revealed that the method is as efficient as the frequency-limited balanced truncation but also more robust to numerical issues which makes it more tractable for ill-conditioned models.

Besides, the proposed algorithm will be soonly made available in the *MORE Toolbox* [14].

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