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$\mathcal{L}_1$ Adaptive Control for Systems with Matched Stochastic Disturbance

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Abstract This paper presents a stochastic state feedback $\mathcal{L}_1$ adaptive control for systems with matched disturbances. The proposed approach is characterized through the introduction of a Kalman type fixed gain in the predictor. The main contribution of this work is that closed loop system analysis is demonstrated through a deterministic-like approach that uses the stochastic Laplace transform. The control is designed to accommodate and to be robust to unknown input gain as well as to system uncertainties. Simulation results show good results for the pitch angle control of a small fixed wing UAV.

1 Introduction

$\mathcal{L}_1$ adaptive control was developed for various classes of uncertain systems [1] and has shown good performance with uncertainties in the plant and external deterministic disturbances. However, in many real situations, disturbances and unmodeled dynamics in physical systems are stochastic. Systems with such random dynamics cannot be handled by deterministic analysis and design approaches. Consequently, dedicated tools are required to treat this problem.

In this paper, is considered a $\mathcal{L}_1$ adaptive control method for systems with matched random disturbances, i. e. systems corrupted by stochastic disturbance which acts in the same direction as the control variable. A Kalman type gain is introduced in the predictor architecture and it is shown that the estimation error is exponentially ultimately bounded in the mean square. A deterministic-like approach based on the stochastic Laplace transform [2, 3] is used to analyze the performance bounds of the system.

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In order to show the application potential of this approach, simulation results of a pitch rate control of a small fixed wing UAV with large uncertainties in aerodynamic parameters are presented.

2 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space and given the following system represented as

$$\begin{aligned} \dot{x}(t) &= A_m x(t) + b(\mu u(t) + \theta^\top x(t) + \sigma(t)), \qquad x(0) = x_0 \\ y(t) &= c^\top x(t) \end{aligned} \tag{1}$$

where $x(t)$ is the $\mathbb{R}^n$-valued solution to (1), $y(t)$ is the $\mathbb{R}$-valued observation of the output of the system, $A_m \in \mathbb{R}^{n\times n}$ is a known Hurwitz matrix that defines the desired dynamics of the system, b, $c \in \mathbb{R}^n$ are known constant vectors, $u(t) \in \mathbb{R}$ the control input, $\theta \in \mathbb{R}^n$ is a vector of constant unknown parameters, $\mu \in \mathbb{R}$ is an unknown constant input gain, $\sigma(t)$ is assumed to be a colored noise, i.e. a linear time invariant system driven by a white noise modeled by

$$\begin{aligned} \dot{x}_\sigma(t) &= A_\sigma x_\sigma(t) + b_\sigma w(t), \qquad x_\sigma(0) = 0_{1\times l} \\ \sigma(t) &= c_\sigma^\top x_\sigma(t) \end{aligned} \tag{2}$$

where $w(t) = w(t,\omega) : [t_0, t_f] \times \Omega \to \mathbb{R}^n$ is assumed to be zero mean Gaussian white noise process with zero mean and variance ξ, $x_\sigma \in \mathbb{R}^l$ is the state vector of the disturbance, $A_\sigma \in \mathbb{R}^{l\times l}$ is a known Hurwitz matrix that defines the dynamics of the disturbance, b_σ and $c_\sigma \in \mathbb{R}^l$ are known constant vectors.

Assumption. The unknown parameter θ is uniformly bounded i.e. $\theta \in \Theta$ where Θ is a known compact convex set, furthermore $L = max_{\theta\in\Theta} \|\theta\|_1$. The unknown input gain μ is partially known, i.e. $\mu \in [\mu_l, \mu_u]$ where $0 < \mu_l < \mu_u$ are given lower and upper bounds of the input gain. The disturbance $\sigma(t)$ is bounded i.e. $|\sigma(t)| < \Delta$ where $\Delta \in \mathbb{R}^+$.

Taking $z(t) = (x(t)\; x_\sigma(t))^\top$ the system (1) can be written in augmented form as

$$\dot{z}(t) = Az(t) + b_u(\mu u(t) + \theta^\top x(t)) + b_w w(t), \qquad z(0) = (x_0\; 0_{1\times l})^\top \tag{3}$$

where $A = \begin{bmatrix} A_m & b\, c_\sigma \\ 0_{d\times n} & A_\sigma \end{bmatrix}$ $b_u = \begin{pmatrix} b \\ 0_{n\times 1} \end{pmatrix}$ and $b_w = \begin{pmatrix} 0_{l\times 1} \\ b_\sigma \end{pmatrix}$

The control objective is to design a state-feedback adaptive controller, such that the system described in (1) follows the desired model given by

$$\begin{aligned} \dot{x}_m(t) &= A_m x_m(t) + b k_g r(t), \qquad x_m(0) = x_0 \\ y_m(t) &= c^\top x_m(t) \end{aligned} \tag{4}$$

where $r(t)$ is is the reference input, $x_m(t)$ is the desired state vector and the static gain k_g is chosen $k_g = -1/(c^\top A_m^{-1} b)$.

3 $\mathcal{L}_1$ Adaptive Controller

Similar to the approach for systems with deterministic uncertainties [1] the proposed approach of $\mathcal{L}_1$ stochastic adaptive control is composed of the state predictor, the adaptation law and the control law Fig. 1.

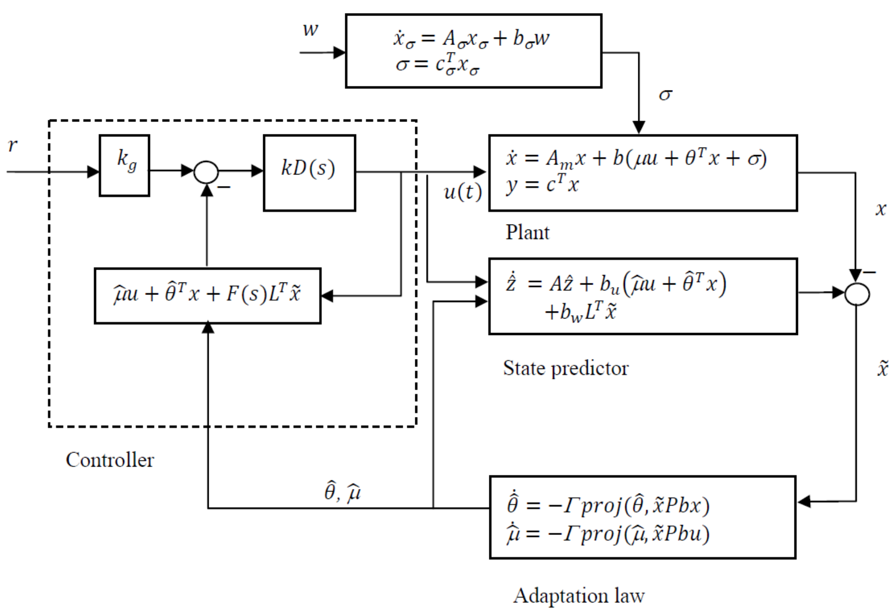

Fig. 1 Block diagram of the control architecture.

The expression of the state predictor, where a Kalman type gain is introduced, is given by

$$\begin{aligned} \dot{\hat{z}}(t) &= A\hat{z}(t) + b_u\left(\hat{\mu}u(t) + \hat{\theta}^\top(t)x(t)\right) + b_w L^\top \tilde{x}(t), \qquad \hat{z}(0) = z_0 \\ \hat{y}(t) &= (c^\top \; 0_{lx1})\hat{z}(t) \end{aligned} \tag{5}$$

where $\hat{z}(t) = \left(\hat{x}(t) \;\; \hat{x}_\sigma(t)\right)^\top$ is the state vector of the predictor, $\tilde{x}(t) = \hat{x}(t) - x(t)$ is the error prediction of the state vector, $\hat{\theta}$ is the estimate of the unknown parameter θ and $L \in \mathbb{R}^n$ is the Kalman type static gain vector.

The control law is given by

$$u(s) = kD(s)(k_g r(s) - \hat{\eta}(s) - F(s)L^\top \tilde{x}(s)) \tag{6}$$

where $\hat{\eta}(s)$ is the Laplace transformation of the term $\hat{\mu}(t)u(t) + \hat{\theta}^\top(t)\hat{x}(t)$, $F(s) = c_\sigma^\top (s\mathbb{I} - A_\sigma)^{-1} b_\sigma$ is the transfer function of the disturbance model and $D(s)$ is a transfer function that leads to a strictly proper stable filter $C(s) = \mu kD(s)/(1 + \mu kD(s))$ with $C(0) = 1$.

The adaptation law is defined by

$$\begin{aligned} \dot{\hat{\mu}}(t) &= \Gamma Proj\left(\hat{\mu}(t), -\tilde{x}^\top(t)Pbu(t)\right) \\ \dot{\hat{\theta}}(t) &= \Gamma Proj\left(\hat{\theta}(t), -\tilde{x}^\top(t)Pbx(t)\right) \end{aligned} \tag{7}$$

where $\Gamma \in \mathbb{R}$ is the adaptation rate, $P = P^\top > 0$ is the solution of the Lyapunov equation $A_m^\top P + PA_m = -Q$ with $Q = Q^\top > 0$ arbitrary and $Proj(\cdot,\cdot)$ is the projection operator described by the following [8]

Definition 1. Suppose that $f(\theta)$:$R^n \to R$ is a continuously differentiable smooth convex function denoted by

$$f(\theta) = \frac{(\varepsilon+1)\theta^\top\theta - \theta_{max}^2}{\varepsilon\theta_{max}^2}$$

with θ_{max} is the norm bound imposed on the vector θ and $\varepsilon > 0$ is an arbitrary tolerance bound. The gradient vector of f evaluated at θ is noted by $\bigtriangledown f(\theta)$.

For a constant $\delta > 0$, consider a convex compact set with a smooth boundary given by

$$\Omega_c = \{\theta \in R^n | f(\theta) < \delta\}$$

the projection operator is defined by

$$Proj(\theta, y) = \begin{cases} y - \frac{\bigtriangledown f(\theta)(\bigtriangledown f(\theta)^\top)}{\|\bigtriangledown f(\theta)\|^2} y f(\theta) & \text{if } f(\theta) > 0 \text{ and } \bigtriangledown f(\theta)^\top y > 0 \\ y & \text{if not} \end{cases}$$

4 Analysis of the Control Architecture

Analysis of the properties of the control architecture involves showing boundedness of the prediction error and demonstrating performance bounds, i. e. the error between reference system and the closed-loop plant adaptive Controller.

4.1 Prediction error dynamics

In this section, the bound the estimation error is shown using mean square stochastic stability, being minimum mean square a frequently used criterion in estimation theory. First, the following definitions are recalled from [4].

Definition 2. Consider the continuous-time stochastic process described by the Itô stochastic differential equation, with a global and unique time-continuous solution

$$dx(t) = f(x,t)dt + g(x,t)dW(t) \tag{8}$$

where $W(t)$ is an independent Wiener process, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any given $V(x) > 0 \in C^2$, associated with the stochastic system (8), the differential operator $\mathcal{L}$ is defined by:

$$\mathcal{L}V(x,t) = \frac{\partial V(x,t)}{\partial x^\top} f(x,t) + \frac{1}{2} Tr \left\{ g^\top(x,t) \frac{\partial^2 V}{\partial x \partial x^\top} g(x,t) \right\} \tag{9}$$

Definition 3. Consider the stochastic differential equation (8). Then $x(t)$ is said to be exponentially ultimately bounded in the mean square if there exist positive constants c_1, c_2 and c_3, such that for all $t \geq 0$ the following expectation is true

$$\mathbb{E}[\|x(t)\|] < c_1 \, e^{-c_2 t} + c_3 \tag{10}$$

where $\mathbb{E}(\cdot)$ denotes the expected value operator.

Next, it is shown in the following theorem, that the prediction error is bounded in mean square.

Theorem 1. The estimation error of the augmented system (3) with the state predictor (5) and the adaptation law (7) is mean-square exponentially ultimately bounded.

Proof. From (3) and (5) the expression of the dynamics of the prediction error can be written in Itô form as

$$d\tilde{z}(t) = \Big(A\tilde{z}(t) + b_u \left(\tilde{\mu} u(t) + \tilde{\theta}^\top(t) x(t) \right) + b_w L^\top \tilde{x}(t) \Big) dt - \xi b_w dW(t) \tag{11}$$

where $\tilde{\mu}(t) = \hat{\mu}(t) - \mu$, $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$ and $W(t)$ is an increment of a Wiener process (Brownian motion) with zero-mean Gaussian distribution and variance given by $\mathbb{E}[dW(t) dW^\top(t)] = 1$.

Under the assumption that the unknown parameters μ and θ are constant, using the adaptation law (7) it can be written

$$\begin{aligned} d\tilde{\mu}(t) &= \Gamma Proj\left(\hat{\mu}(t), -\tilde{x}(t)Pbu(t)\right) dt \\ d\tilde{\theta}(t) &= \Gamma Proj\left(\hat{\theta}(t), -\tilde{x}(t)Pbx(t)\right) dt \end{aligned} \tag{12}$$

Taking $\zeta(t) = \left(\tilde{x}(t) \;\; \tilde{x}_\sigma(t) \;\; \tilde{\mu}(t) \;\; \tilde{\theta}(t) \right)^\top$ equations (11) and (12) are written

$$d\zeta(t) = f(\zeta,t)dt + g(\zeta,t)dW \tag{13}$$

where $f(\zeta,t) = \begin{pmatrix} A_m\tilde{x}(t) + b\left(\tilde{\mu}u(t) + \tilde{\theta}^\top(t)x(t) + c_\sigma^\top \tilde{x}_\sigma(t)\right) \\ A_\sigma \tilde{x}_\sigma + b_\sigma L^\top \tilde{x}(t) \\ \Gamma Proj\left(\hat{\mu}(t), -\tilde{x}(t)Pbu(t)\right) \\ \Gamma Proj\left(\hat{\theta}(t), -\tilde{x}(t)Pbx(t)\right) \end{pmatrix}$ and $g(\zeta,t) = \begin{pmatrix} 0_{n\times 1} \\ -\xi b_\sigma \\ 0 \\ 0 \end{pmatrix}$

Considering the Lyapunov function candidate

$$V(\zeta,t) = \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{x}_\sigma^\top(t)P_\sigma\tilde{x}_\sigma(t) + \Gamma^{-1}\left(\tilde{\theta}^\top(t)\tilde{\theta}(t) + \tilde{\mu}^2(t)\right) \tag{14}$$

with $P_\sigma = P_\sigma^\top > 0$ is the solution of the Lyapunov equation $A_\sigma^\top P_\sigma + P_\sigma A_\sigma = -Q_\sigma$ with $Q_\sigma = Q_\sigma^\top > 0$ arbitrary. Thus, the expression of the differential generator of $V(\zeta,t)$ is written

$$\begin{aligned} \mathcal{L}V(\zeta,t) =& \tilde{x}^\top(t)(PA_m + A_m^\top P)\tilde{x}(t) + 2\tilde{x}^\top(t)Pb\tilde{\mu}u(t) + 2\tilde{x}^\top(t)Pb\tilde{\theta}^\top(t)x(t) \\ &+ 2\tilde{x}^\top(t)Pbc_\sigma^\top\tilde{x}_\sigma(t) + \tilde{x}_\sigma^\top(t)(P_\sigma A_\sigma + A_\sigma^\top P_\sigma)\tilde{x}_\sigma(t) \\ &+ 2\tilde{x}_\sigma^\top(t)P_\sigma b_\sigma L^\top\tilde{x}(t) + 2\tilde{\mu}(t)Proj\left(\hat{\mu}(t), -\tilde{x}(t)Pbu(t)\right) \\ &+ 2\tilde{\theta}^\top(t)Proj\left(\hat{\theta}(t), -\tilde{x}(t)Pbx(t)\right) \\ &+ \frac{1}{2}\xi^2 Tr\left\{b_\sigma^\top P_\sigma b_\sigma\right\} \\ =& -\tilde{x}^\top(t)Q\tilde{x}(t) - \tilde{x}_\sigma^\top(t)Q_\sigma\tilde{x}_\sigma + 2\tilde{x}^\top(t)\left(Pbc_\sigma^\top + Lb_\sigma^\top P_\sigma\right)\tilde{x}_\sigma(t) \\ &+ 2\tilde{\mu}(t)\left(\tilde{x}^\top(t)Pbu(t) + Proj(\hat{\mu}(t), -\tilde{x}(t)Pbu(t))\right) \\ &+ 2\tilde{\theta}^\top(t)\left(x(t)\tilde{x}^\top(t)Pb + Proj(\tilde{\theta}(t), -\tilde{x}(t)Pbx(t))\right) \\ &+ \frac{1}{2}\xi^2 Tr\left\{b_\sigma^\top P_\sigma b_\sigma\right\} \end{aligned} \tag{15}$$

Given the adaptation law in (7) one can derive the following bound

$$\begin{aligned} \mathcal{L}V(\zeta,t) \leq & -\lambda_{min}(Q)\tilde{x}^\top(t)\tilde{x}(t) - \lambda_{min}(Q_\sigma)\tilde{x}_\sigma^\top(t)\tilde{x}_\sigma(t) \\ &+ 2\tilde{x}^\top(t)\left(Pbc_\sigma^\top + Lb_\sigma^\top P_\sigma\right)\tilde{x}_\sigma(t) + \frac{1}{2}\xi^2 Tr\left\{b_\sigma^\top P_\sigma b_\sigma\right\} \end{aligned} \tag{16}$$

where $\lambda_{max}(\cdot)$, $\lambda_{min}(\cdot)$ are respectively the maximum/minimum eigenvalue of a matrix.

Choosing $L = -Pbc_\sigma^\top P_\sigma b_\sigma (b_\sigma^\top P_\sigma^2 b_\sigma)^{-1}$ enables the elimination of the third right hand term of inequality (16). Furthermore, for simplicity, the arbitrary matrices Q_σ and Q can be chosen such that $\lambda_{min}(Q_\sigma) = \lambda_{min}(Q)$ and consequently

$$\mathcal{L}V(\zeta,t) \leq -\lambda_{min}(Q)\|\tilde{z}\|^2 + \frac{1}{2}\xi^2 Tr\left\{b_\sigma^\top P_\sigma b_\sigma\right\} \tag{17}$$

Moreover, since the projection-based adaptation law ensures that $\hat{\theta}(t) \in \Theta$ and $\mu_l \leq \hat{\mu}(t) \leq \mu_u$, hence the Lyapounov function $V(\zeta,t)$ in (14) can be bounded as

$$\begin{aligned} V(\zeta,t) &\leq \tilde{x}^\top(t)P\tilde{x}(t) + \tilde{x}_\sigma^\top(t)P_\sigma\tilde{x}_\sigma(t) + \frac{1}{\Gamma}\left(4\theta_{max}^2 + (\mu_u - \mu_l)^2\right) \\ &:\leq max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]\|\tilde{z}\|^2 + \frac{1}{\Gamma}\left(4\theta_{max}^2 + (\mu_u - \mu_l)^2\right) \end{aligned} \tag{18}$$

where $\theta_{max} = max_{\theta\in\Theta}\|\theta\|^2$. Further given

$$\lambda_{min}(Q)\|\tilde{z}\|^2 = \frac{\lambda_{min}(Q)}{max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]} max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]\|\tilde{z}\|^2 \tag{19}$$

it follows that

$$\lambda_{min}(Q)\|\tilde{z}\|^2 \geq \frac{\lambda_{min}(Q)}{max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]}\left(V(\zeta,t) - \frac{1}{\Gamma}\left(4\theta_{max}^2 + (\mu_u - \mu_l)^2\right)\right) \tag{20}$$

and thus the upper bound in (17) can be used to obtain

$$\mathcal{L}V(\zeta,t) \leq -k_1V(\zeta,t) + k_2 \tag{21}$$

where

$$k_1 = \frac{\lambda_{min}(Q)}{max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]}$$

and

$$k_2 = \frac{1}{\Gamma}\frac{\lambda_{min}(Q)\left(4\theta_{max}^2 + (\mu_u - \mu_l)^2\right)}{max\left[\lambda_{max}(P), \lambda_{max}(P_\sigma)\right]} + \frac{1}{2}\xi^2 Tr\left\{b_\sigma^\top P_\sigma b_\sigma\right\}$$

From [5], it follows that

$$\begin{aligned} \mathbb{E}\left[V(\zeta,t)\right] &\leq \nu_0\, e^{-k_1t} + \frac{k_2}{k_1}\left(1 - e^{-k_1t}\right) \\ &:\leq \left(\nu_0 - \frac{k_2}{k_1}\right)e^{-k_1t} + \frac{k_2}{k_1} \end{aligned} \tag{22}$$

where $\nu_0 = \mathbb{E}\left[V(\zeta,0)\right]$.

Given

$$min\left(\lambda_{min}(P), \lambda_{min}(P_\sigma), \Gamma^{-1}\right)\|\zeta\|^2 \leq V(\zeta,t) \tag{23}$$

it can be written

$$\mathbb{E}\left[\|\zeta\|^2\right] \leq \mathbb{E}\left[V(\zeta,t)\right]/min\left(\lambda_{min}(P), \lambda_{min}(P_\sigma), \Gamma^{-1}\right) \tag{24}$$

and consequently the prediction error $\zeta = [\tilde{x}(t),\, \tilde{x}_\sigma(t),\, \tilde{\mu}(t)$ and $\tilde{\theta}(t)]^\top$ is exponentially ultimately bounded in mean square and the proof is complete □

Lemma 1. The following bound holds, almost surely, for the prediction error of state vector of the plant

$$\|\tilde{x}(t)\| \leq \rho(t), \ \ \forall t \geq 0$$

where

$$\rho(t) = \frac{1}{\sqrt{\lambda_{min}(P)}} \sqrt{\frac{k_2}{k_1}\left(2e^{k_1\delta} - 1\right)e^{\varepsilon t} + e^{2k_1\delta}\left(\nu_0 - \frac{k_2}{k_1}\right)e^{-(k_1-2\varepsilon)t}}$$

where $\delta > 0$, and $\varepsilon \in (0, k_1/2)$ are arbitrary constants. Furthermore, we have the Lyapunov exponent

$$\limsup_{t\to\infty} \frac{1}{t}\log\|\tilde{x}(t)\| \leq 0$$

Proof. If equation (21) is verified, thus using [7] theorem 7, it follows that for the arbitrary constants δ, and ε defined above, there exists a random instant t_0 such that for all $t > t_0$, we have

$$\lambda_{min}(P)\|\tilde{x}(t)\|^2 \leq V(\zeta,t) \leq \frac{k_2}{k_1}\left(2e^{k_1\delta} - 1\right)e^{\varepsilon t} + e^{2k_1\delta}\left(\nu_0 - \frac{k_2}{k_1}\right)e^{-(k_1-2\varepsilon)t} \quad (25)$$

Furthermore, given $\lambda_{min}(P)\|\tilde{x}(t)\|^2 \leq V(\zeta,t)$, thus for $t > 0$ the following inequality holds

$$\frac{1}{t}\log\left(\lambda_{min}(P)\|\tilde{x}(t)\|^2\right) \leq \frac{1}{t}\log(V(\zeta,t)) \quad (26)$$

and using here again [7] theorem 7 it follows that

$$\limsup_{t\to\infty}\frac{1}{t}\log\|\tilde{x}(t)\| \leq \limsup_{t\to\infty}\frac{1}{t}\log(V(\zeta,t)) \leq 0 \quad (27)$$

and the proof is complete □

Remark 1. Note that choosing high adaptation gain, contribute to the optimization of the bound of the prediction error $\|\tilde{x}(t)\|$. Indeed minimizing the factor k_2/k_1 leads to an optimal bound of the estimation error.

Given

$$\frac{k_2}{k_1} = \frac{1}{\Gamma}\left(4\theta_{max}^2 + (\mu_u - \mu_l)^2\right) + \frac{1}{2}\xi^2 Tr\{b^\top Pb\}\frac{max[\lambda_{max}(P), \lambda_{max}(P_\sigma)]}{\lambda_{min}(Q)}$$

For high adaptation gain, this factor can be approximated to become

$$\frac{k_2}{k_1} \simeq \frac{1}{2}\xi^2 Tr\{b^\top Pb\}\frac{max[\lambda_{max}(P), \lambda_{max}(P_\sigma)]}{\lambda_{min}(Q)}$$

solving this problem of optimization by LMI methods as in [7] will lead to an optimal almost sure bound of the prediction error.

4.2 Closed loop reference system

In this section, the reference system, i.e. the closed loop system with nominal parameters, is introduced and its stability is shown through the use of stochastic Laplace transform [2, 3]. Stochastic Laplace transform is an extension of the theory of Laplace transforms in the context of the Itô-Doob stochastic calculus. This method provides an algebraic approach for finding Itô-Doob type stochastic integrals and solving stochastic linear differential equations of the Itô-Doob type.

Definition 4. Let $g(t,W(t))$ be a real valued function of two variables $(t,W(t))$ defined for all real numbers $t \geq 0$ and $W(t)$ be a Wiener process. The Laplace transform of g in the sense of the Itô-Doob integral or stochastic Laplace transform is denoted by

$$G^W(s) = L^W(g(t,W(t))) = \int_{t=0}^{\infty} e^{-st} g(t,W(t))dW(t) \tag{28}$$

for all values of s for which this improper integral exists.

Note that the stochastic Laplace transform inherits linearity, derivative, integral and convolution properties of deterministic Laplace transforms [2, 3].

Next, in order to derive the dynamics of the reference system of the plant, the case of known parameters is considered and it is written as

$$\begin{aligned} x_{ref}(t) &= A_m x_{ref} + b(\mu u_{ref}(t) + \theta^\top x_{ref}(t) + \sigma(t)) \quad x_{ref}(0) = x_0 \\ y_{ref}(t) &= c^\top x_{ref} \end{aligned} \tag{29}$$

The control law is given by

$$u_{ref}(s) = \frac{C(s)}{\mu}\left(k_g r(s) - \theta^\top x_{ref}(s) - F(s)K^\top \tilde{x}_{ref}(s)\right) \tag{30}$$

Defining proper BIBO stable transfer functions $H(s) = (s\mathbb{I} - A_m)^{-1}b$ and $G(s) = H(s)(1 - C(s))$, the stability of the closed loop reference system is demonstrated through the following lemma.

Lemma 2. If the filter $C(s)$ is designed such that the $\mathcal{L}_1$ norm condition $\|G(s)\|_{\mathcal{L}_1} L < 1$ [1] is verified, then the closed-loop reference system in (29) and (30) is BIBS stable with respect to the reference input and initial conditions.

Proof. From (2) the expression of the stochastic disturbance is written in Itô form as

$$\begin{aligned} dx_\sigma(t) &= A_\sigma x_\sigma dt + b_\sigma \xi dW(t) \\ \sigma(t) &= c_\sigma x_\sigma(t) \end{aligned} \tag{31}$$

writing (31) as an integral equation leads to

$$x_\sigma(t) = \int_0^t A_\sigma x_\sigma(\tau)d\tau + b_\sigma \xi \int_0^t dW(\tau) \tag{32}$$

Using the symbol L as a Laplace operator and taking the Laplace transformation of (32) it can be written

$$\begin{aligned} X_\sigma(s) =& L\left(\int_0^t A_\sigma x_\sigma(\tau)d\tau + \xi b_\sigma \int_0^t dW(\tau)\right) \\ =& A_\sigma L\left(\int_0^t x_\sigma(\tau)d\tau\right) + b_\sigma \xi L\left(\int_0^t dW(\tau)\right) \end{aligned} \tag{33}$$

Using properties of the stochastic Laplace transform [2, 3] it can be written

$$L(dW(t)) = L^W(1) \tag{34}$$

and consequently

$$X_\sigma(s) = \frac{1}{s} A_\sigma X_\sigma(s) + b_\sigma \xi \frac{L^W(1)}{s} \tag{35}$$

which leads to

$$(s\mathbb{I} - A_\sigma)X_\sigma(s) = b_\sigma \xi L^W(1) \tag{36}$$

Hence, the Laplace transformation of the stochastic disturbance $\sigma(t)$ is written

$$\Sigma(s) = c_\sigma^\top X_\sigma(s) = \xi F(s) L^W(1) \tag{37}$$

Consequently, the closed loop reference system is written

$$\begin{aligned} x_{ref}(s) =& G(s)\theta^\top x_{ref}(s) + C(s)H(s)k_g r(s) \\ & - C(s)H(s)F(s)K^\top \tilde{x}_{ref}(s) + \Sigma(s) + x_{in}(s) \end{aligned} \tag{38}$$

where $x_{in}(s) = (s\mathbb{I} - A_m)^{-1}x_0$. Thus, from [1] lemma A.7.1 it follows that for all $\tau \geq 0$ the following bound holds

$$\begin{aligned} \left\|x_{ref\ \tau}\right\|_{\mathcal{L}_\infty} \leq & \left\|G(s)\theta^\top\right\|_{\mathcal{L}_1} \left\|x_{ref\ \tau}\right\|_{\mathcal{L}_\infty} + \left\|C(s)H(s)k_g\right\|_{\mathcal{L}_1} \left\|r_\tau\right\|_{\mathcal{L}_\infty} \\ & + \left\|C(s)H(s)F(s)K^\top\right\|_{\mathcal{L}_1} \left\|\tilde{x}_{ref\ \tau}\right\|_{\mathcal{L}_\infty} + \left\|\sigma_\tau\right\|_{\mathcal{L}_1} + \left\|x_{in}\right\|_{\mathcal{L}_\infty} \end{aligned} \tag{39}$$

Given the condition of $\mathcal{L}_1$ stability [1] equation (39) is written

$$\begin{aligned} \left\|x_{ref\ \tau}\right\|_{\mathcal{L}_\infty} \leq & \frac{\left\|C(s)H(s)k_g\right\|_{\mathcal{L}_1} \left\|r_\tau\right\|_{\mathcal{L}_\infty}}{1 - \left\|G(s)\right\|_{\mathcal{L}_1} L} \\ & + \frac{\left\|C(s)H(s)F(s)K^\top\right\|_{\mathcal{L}_1} \left\|\tilde{x}_{ref\ \tau}\right\|_{\mathcal{L}_\infty} + \Delta + \left\|x_{in}\right\|_{\mathcal{L}_\infty}}{1 - \left\|G(s)\right\|_{\mathcal{L}_1} L} \end{aligned} \tag{40}$$

Since $r(t), x_{in}(t)$, $\psi(t)$ and $\tilde{x}_{ref}(t)$ are bounded it is straightforward that the reference state $x_{ref}(t)$ in (40) is bounded and the proof is complete. □

4.3 Performance Bounds

The following theorem states on the transient performances of the closed loop system i. e. the tracking errors between the reference system and the plant with $\mathcal{L}_1$ adaptive control, and it is shown that the transient regime is strongly connected to the estimation error.

Theorem 2. Given the closed loop system (1), (6) and the reference system (29) (30), the following bound holds

$$\left\|x(t)-x_{ref}(t)\right\|_\infty \leq \frac{2}{\sqrt{\lambda_{min}(P)}}\left(\varphi_{H_1}(t)*\rho(t)\right)$$
$$\left\|u(t)-u_{ref}(t)\right\|_\infty \leq \frac{2}{\sqrt{\lambda_{min}(P)}}\left(\varphi_{H_2}(t)*\rho(t)\right)$$

where $\varphi_e(t)=\max_{i=1,\ldots,n}\sqrt{\sum_{j=1}^m e_{ij}^2(t)}$, e_{ij} (t) is the ith row, jth column of the impulse response matrix of $E(s)$ [1] and $H_1(s)$, $H_2(s)$ are defined below.

Proof. The control law in (6) can be written as

$$u(s)=\frac{C(s)}{\mu}\left(k_g r(s)-\theta^\top x_{ref}(s)-F(s)K^\top\tilde{x}_{ref}(s)-\tilde{\eta}(s)\right) \tag{41}$$

where $\tilde{\eta}(s)$ is the Laplace transform of the term $\tilde{\mu}(t)u(t)+\tilde{\theta}^\top(t)x(t)$. The closed loop system takes the form

$$x(s)=G(s)\theta^\top x(s)+C(s)H(s)\left(k_g r(s)-F(s)K^\top\tilde{x}(s)-\tilde{\eta}(s)\right)+\Sigma(s)+x_{in}(s) \tag{42}$$

From (38) it follows that

$$\begin{aligned} x_{ref}(s)-x(s)=&G(s)\theta^\top\left(x_{ref}(s)-x(s)\right)\\ &-C(s)H(s)\left(F(s)K^\top\left(\tilde{x}_{ref}(s)-\tilde{x}(s)\right)+\tilde{\eta}(s)\right) \end{aligned} \tag{43}$$

Given (11) the Laplace transform of error dynamics of the plant $\tilde{x}(t)$ and the reference $\tilde{x}_{ref}(t)$ is written

$$\begin{aligned} \tilde{x}(s)&=H(s)\tilde{\eta}(s)+H(s)F(s)K^\top\tilde{x}(s)-\Sigma(s)\\ \tilde{x}_{ref}(s)&=H(s)F(s)K^\top\tilde{x}_{ref}(s)-\Sigma(s) \end{aligned} \tag{44}$$

Replacing (44) in (43) leads to

$$x_{ref}(s) - x(s) = H_1(s)\left(\tilde{x}_{ref}(s) - \tilde{x}(s)\right) \leq 2H_1(s)\,\rho(s) \tag{45}$$

where $H_1(s) = -(\mathbb{I} - G(s)\theta^\top)^{-1}C(s)$. Using the same approach in [1] lemma 2.2.6, (45) leads to the bound of the state vector.

To show the bound of the control law, from equations (6) and (30) it is written

$$u_{ref}(s) - u(s) = -\frac{C(s)}{\mu}\left(\theta^\top\left(x_{ref}(s) - x(s)\right) + \left(K^\top\left(\tilde{x}_{ref}(s) - \tilde{x}(s)\right) - \tilde{\eta}(s)\right)\right) \tag{46}$$

From Lemma A.12.1 in [1] there exists a vector c_0 such that

$$\begin{aligned} u_{ref}(s) - u(s) = &-\frac{C(s)}{\mu}\theta^\top(x_{ref}(s) - x(s)) \\ &- \frac{C(s)}{\mu}\frac{1}{c_0^\top H(s)}c_0^\top H(s)\left(F(s)K^\top\left(\tilde{x}_{ref}(s) - \tilde{x}(s)\right) - \tilde{\eta}(s)\right) \end{aligned} \tag{47}$$

From (44) and it follows hat

$$\begin{aligned} u_{ref}(s) - u(s) = &-\frac{C(s)}{\mu}\theta^\top(x_{ref}(s) - x(s)) \\ &- \frac{C(s)}{\mu}\frac{1}{c_0^\top H(s)}c_0^\top\left(\tilde{x}_{ref}(s) - \tilde{x}(s)\right) \end{aligned} \tag{48}$$

There exists also a vector c_1 such that

$$\begin{aligned} u_{ref}(s) - u(s) = &-\frac{C(s)}{\mu}\theta^\top(x_{ref}(s) - x(s)) \\ &- \frac{C(s)}{\mu}\frac{1}{c_0^\top H(s)}c_0^\top\frac{1}{c_1^\top H_1(s)}c_1^\top H_1(s)\left(\tilde{x}_{ref}(s) - \tilde{x}(s)\right) \end{aligned} \tag{49}$$

Defining $H_2(s) = -\frac{C(s)}{\mu}\left(\theta^\top + \frac{1}{c_0^\top H(s)}c_0^\top\frac{1}{c_1^\top H_1(s)}c_1^\top\right)$ leads to

$$u_{ref}(s) - u(s) = H_2(s)\left(x_{ref}(s) - x(s)\right) \tag{50}$$

Using the same approach in [1] lemma 2.2.6, equation (50) leads to the expression of the bound of the control law and completes the proof. □

5 Simulation results

The control law proposed and analyzed in previous sections is now applied to the pitch dynamics of a fixed wing UAV, a model of the Monsun BO 209. Tracking performance under stochastic matched disturbance and model uncertainties is shown through simulations.

The short period dynamics can be written in matrix form as in [9]

$$\underbrace{\begin{pmatrix} \dot{\alpha} \\ \dot{q} \end{pmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} \frac{Z_\alpha}{V_a} & 1+\frac{Z_q}{V_a} \\ M_\alpha & M_q \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} \alpha \\ q \end{pmatrix}}_{x} + \underbrace{\begin{pmatrix} \frac{Z_\delta}{V_a} \\ M_\delta \end{pmatrix}}_{b} \underbrace{\delta_e}_{u}$$

$$y = q = \underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix}}_{c^\top} x$$

where α is the angle of attack, q is the pitch angular velocity, V_a is the trimmed airspeed, $(Z_\alpha,\ Z_q, Z_\delta)$ and $(M_\alpha, M_q, M_\delta)$ are the partial derivatives of the aerodynamic force Z and the pitching moment M, with respect to α, q, and δ_e, respectively. The control input is the elevator angle δ_e.

Numerical values for the considered UAV trimmed at $V_{a0} = 20\ m/s$, $\alpha_0 = 4°$, $q_0 = 0°/s$, and altitude h_0= 50 m are given by

$$A = \begin{bmatrix} -11.0447 & 0.9644 \\ -242.4575 & -14.4717 \end{bmatrix}, \quad b = \begin{pmatrix} -0.2840 \\ -112.4126 \end{pmatrix}.$$

Note that actuator dynamics are assumed to be negligible.

The system is affected by an additive matched disturbance $\sigma(t)$ with dynamics $F(s) = 1/s+1$ and input $w(t)$ as a white noise with variance $\xi = 1$.

The desired dynamics matrix A_m for system (1) is chosen such that it meets military specifications for category A, level-1 flight handling qualities requirement and eigenvalues for the short period mode are given from [9] as $\lambda_{1,2} = -5.6 \pm 4.2j$, i. e. a pulsation $\omega_n = 7 rad/s$ and a damping $\varsigma = 0.8$.

$\mathcal{L}_1$ adaptive controller parameters are set $\Gamma = 100000$, $D(s) = 1/s$ and $k = 2000$. Note however that even if this value of k is high it affects only the filter $C(s)$ and it has no effect on stability margins of the controller [10].

The control approach is based on the augmentation of a linear controller by the $\mathcal{L}_1$ adaptive controller, as the common approach in aerospace systems. This allows the use of the available knowledge about the system dynamics. The adaptive controller is then added to compensate unknown parameters and / or disturbances effect.

The controller is designed to be robust against uncertainties and the compact sets are set to $\mu = (0.41.4)$, $\Theta = \left\{\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{R}^2 : \vartheta_i \in (-2,2), i = 1,2\right\}$. From the definition of Θ it follows that $L = 2$, which results in $\|G(s)L\|_1 = 0.2277$ which satisfies the $\mathcal{L}_1$ stability condition.

Fig. 2 depicts the response of the system to a square signal reference. It can be seen that the control architecture reduces the effect of noise disturbance.

It is noted in Fig. 3 that the prediction error of the state vector is bounded and negligible, thus it can be concluded that the predictor is stable and it presents good prediction performance.

To show pertinence of the developed controller, failures are introduced to the system as a loss of actuator effectiveness $\mu = 0.5$ and the UAV becomes marginally stable $M_q = 0$, and statically unstable $M_\alpha > 0$. Such drastic and perhaps unrealistic

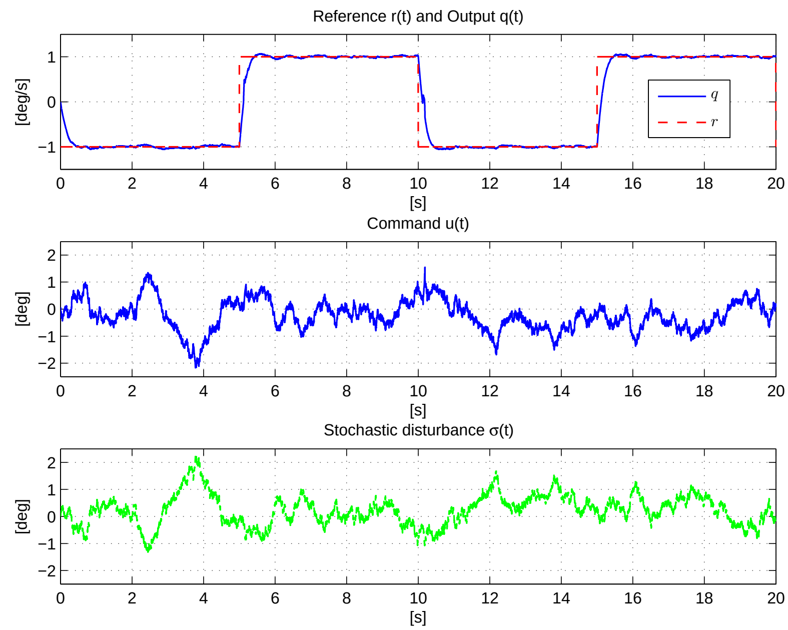

Fig. 2 Output of the closed loop system for system without failures.

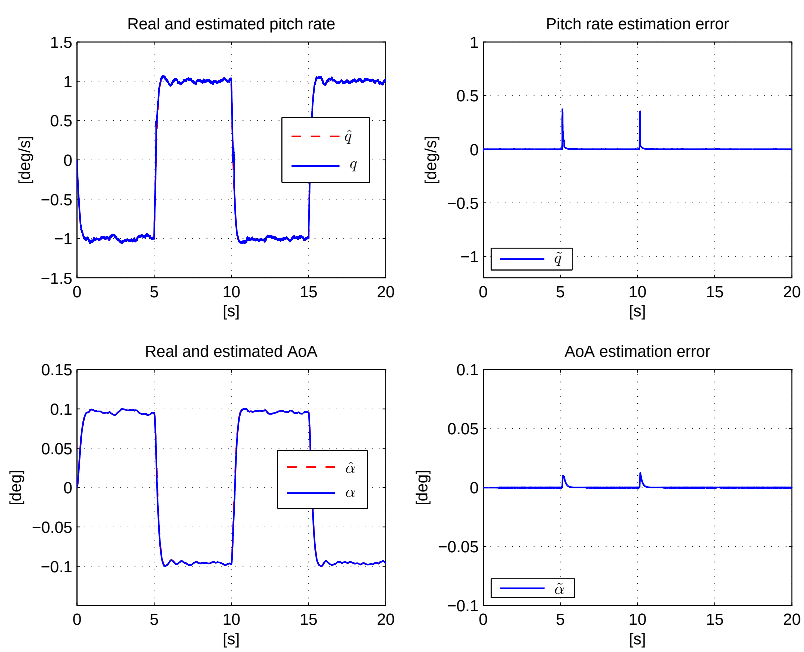

Fig. 3 Estimation performance of the system without failures.

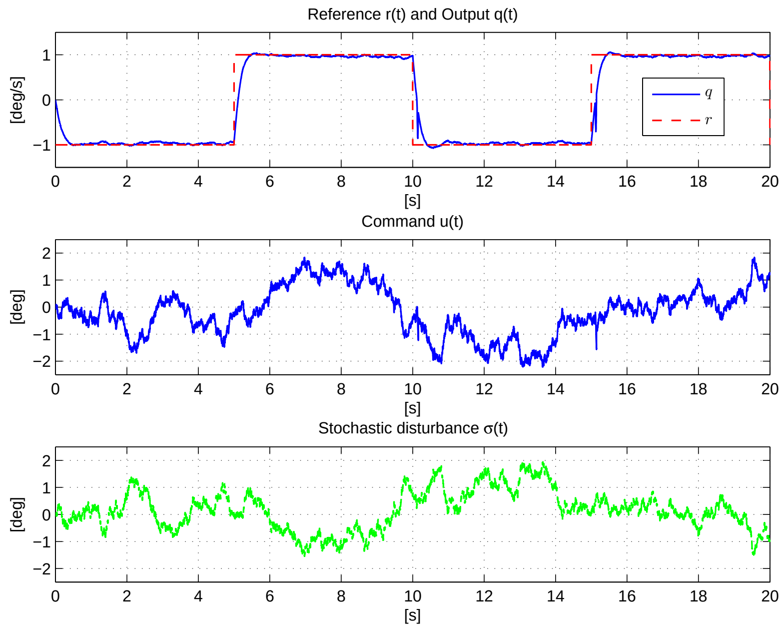

Fig. 4 Output of the closed loop system for system with failures.

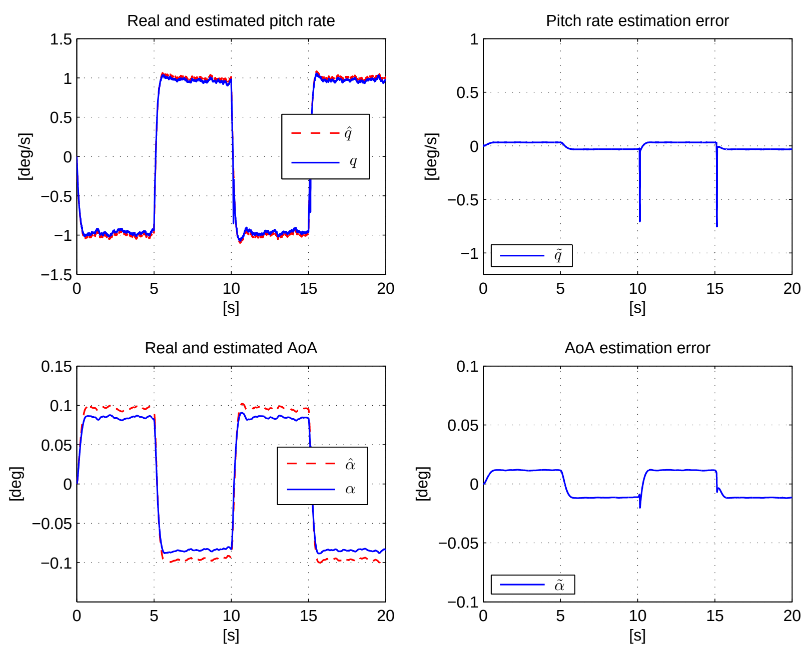

Fig. 5 Estimation performance of the system with failures.

situation is motivated by the intent to demonstrate the effectiveness of the proposed approach. Fig. 4 and Fig. 5 show that even under large uncertainties, the system has good performance and compensates disturbance effect. Moreover, the elevator command is within acceptable limits.

6 Conclusion

In this paper, a $\mathcal{L}_1$ adaptive control scheme for systems with matched stochastic uncertainties has been proposed. A Kalman type gain is introduced in the predictor architecture and it is shown that the estimation error is exponentially ultimately bounded in the mean square. Closed loop boundedness and performance are analyzed using the stochastic Laplace transform. This permits the use of an analysis approach similar to systems with deterministic disturbances.

Simulation results showed good performances for pitch angle control of a small fixed wing UAV in the presence of strong disturbances (such as air turbulences).

This work is a starting point on the application of $\mathcal{L}_1$ adaptive approach to the control of stochastic systems and several areas of investigation remain for this proposed control method, including non matched disturbances, output-feedback systems with measurement noise. Furthermore, the use of stochastic Laplace transform opens a lot of perspectives for random systems control and analysis.

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