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The Influence of the Taylor Series Remainder on an Incremental Non-linear Dynamic Inversion Controller

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Abstract This paper presents an analysis of a non-linear control algorithm called incremental non-linear dynamic inversion. A Taylor series approximation is used in this algorithm, neglecting higher order terms. This could destabilize the controller if the error made is not bounded. By making use of the Taylor series remainder term and the bounding properties it has, a derivation is made showing that the control algorithm is able to reject these inaccuracies under certain conditions. It is also shown that the incremental non-linear dynamic inversion controller remains robust towards model uncertainties under the influence of the remainder.

1 Introduction

Incremental Non-linear Dynamic Inversion or INDI for short has many advantages over the related Non-linear Dynamic Inversion (NDI), see [1], [2], [3] and [4]. One of the greatest advantages is that it is not required to have an accurate (aerodynamic) model. One of the disadvantages is that in the derivation of the INDI algorithm a Taylor Series approximation is used. In this approximation higher order terms are neglected inducing possible errors. Following is a quantization of this error and statements about the influence it has on the accuracy of this control algorithm.

First the remainder of the Taylor series will be derived for both one dimensional and multi-variable functions. Next the INDI algorithm is derived in the case model uncertainties are present and including the influence of the remainder. An example model will be analyzed using the just derived controller and remainder function. The last section states the conclusions

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2 Taylor Series Approximation

It is very common to use a Taylor series approximation when linearizing a non-linear system despite the error made by neglecting higher order terms. A magnitude for this error is called the remainder $\mathbf{R}_\mathrm{n}$ of which n equals to the order of the approximation. We will first state this value for a Taylor Series depending on just one variable, followed by the derivation of the remainder of a multi-variable function.

2.1 Single variable remainder

For a function $f : \mathbf{R} \to \mathbf{R}$, it can be stated that:

$$f(x) = \sum_{\mathrm{n}=0}^{k} \frac{f^{(\mathrm{n})}(a)}{\mathrm{n}!}(x-a)^{\mathrm{n}} + \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \tag{1}$$

in which the Taylor series equals to:

$$T_k(x) = \sum_{\mathrm{n}=0}^{k} \frac{f^{(\mathrm{n})}(a)}{\mathrm{n}!}(x-a)^{\mathrm{n}} \tag{2}$$

and the remainder to:

$$R_k(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \tag{3}$$

For this to be true it is required that $f^{(n+1)}(x)$ is continues on an open interval I, also I must contain a, and x is in I. As n approaches infinity, $\mathrm{n} \to \infty$, the remainder will approach zero:

$$\lim_{k\to\infty} R_k(x) = 0 \tag{4}$$

Continuing with Eq. (3) the one dimensional Taylor's inequality theorem is proven in [5].

Theorem 1 (Taylor's Inequality).

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \tag{5}$$

in which

$$|f^{(k+1)}(x)| \leq M \tag{6}$$

and

$$|x-a| \leq d \tag{7}$$

From this theorem it is clear that the remainder is bounded but depends on the order of the approximation k and the value of d. It is interesting to note that for any value of $f^{(k+1)}$ the remainder can be reduced by decreasing d. Also one should note that in practise the remainder might be negative.

2.2 Multi-variable remainder

The approximation of the function $f(x)$ as given in Eq. (1) is one dimensional. To extend theorem 1 for a function $f(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$ and later $\mathbf{f}(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}^n$, a coordinate transformation can be used. In the case both $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{a} \in \mathbf{R}^n$ lay in a compact set and $f(\mathbf{x})$ is continues on the considered closed interval, the Taylor series follows a linear path between $\mathbf{x}$ and $\mathbf{a}$. For this a function $\mathbf{u}(t) : \mathbf{R} \to \mathbf{R}^n$ is defined which only depends on the variable $t \in \mathbf{R}$:

$$\mathbf{u}(t) = \mathbf{a} + t(\mathbf{x} - \mathbf{a}) \tag{8}$$

next this function is applied in

$$g(t) = f(\mathbf{u}(t)) \tag{9}$$

defining the function $g(t)$ which can replace the function $f(t)$ in Eq. (1) and integrating over t on the closed interval $[0,1]$:

$$\begin{aligned} f(\mathbf{x}) = g(1) &= g(0) + \sum_{\mathrm{n}=1}^{k} \frac{g^{(\mathrm{n})}(0)}{\mathrm{n}!}(1-0)^{\mathrm{n}} + \frac{1}{k!}\int_0^1 (1-t)^k g^{(k+1)}(t)\mathrm{d}t \\ &= g(0) + \sum_{\mathrm{n}=1}^{k} \frac{g^{(\mathrm{n})}(0)}{\mathrm{n}!} + \frac{1}{k!}\int_0^1 (1-t)^k g^{(k+1)}(t)\mathrm{d}t \end{aligned} \tag{10}$$

The next step is to obtain the derivatives of the function $g(t)$:

$$\begin{aligned} g^{(j)}(t) &= \frac{\mathrm{d}^j}{\mathrm{d}t^j} f(\mathbf{u}(t)) \\ &= \frac{\mathrm{d}^j}{\mathrm{d}t^j} f(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) \\ &= \sum_{|\alpha|=j} \binom{j}{\alpha} (D^\alpha f)(\mathbf{a} + t(\mathbf{x} - \mathbf{a})(\mathbf{x} - \mathbf{a})^\alpha \end{aligned} \tag{11}$$

the chain rule has been applied to $f(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))$ and a multi-index notation is used to express the factorial and partial derivatives. For the multi-index notation a n-tuple is used:

$$\alpha = (\alpha_1,\ \alpha_2, \cdots, \alpha_n) \tag{12}$$

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{13}$$

$$\binom{j}{\alpha} = \frac{j!}{\alpha_1!\alpha_2!\cdots\alpha_n!} = \frac{j!}{\alpha!} \tag{14}$$

$$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n! \tag{15}$$

With the partial derivatives written as:

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \tag{16}$$

When using Eq. (11) in Eq. (10) the Taylor series approximation for a multi-variable function $f(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}$ is obtained:

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{a}) &+ \sum_{|\alpha|=1}^{k} \frac{1}{\alpha!}(D^\alpha f)(\mathbf{a})(\mathbf{x}-\mathbf{a})^\alpha \\ &+ \sum_{|\beta|=k+1} \frac{k+1}{\beta!}\int_0^1 (1-t)^k (D^\beta f)(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})^\beta \mathrm{d}t \end{aligned} \tag{17}$$

To clarify the multi-index notation one can take the example of a first order Taylor series of the function $f(\mathbf{x}) : \mathbf{R}^2 \to \mathbf{R}$. In the case of a first order expansion k and $|\alpha|$ are equal to one. The expression $\alpha!$ is also equal to one on both cases of $\alpha = (\alpha_1,\, \alpha_2) = (1,\, 0)$ or $(0,\, 1)$. The values of $|\beta|$ and $k+1$ are equal to two. This results in three combinations for β: (2, 0), (0, 2) and (1, 1), and three partial derivatives: $\frac{\partial^2}{\partial {x_1}^2}$, $\frac{\partial^2}{\partial {x_2}^2}$ and $\frac{\partial^2}{\partial x_1 \partial x_2}$.

Now that the expression of the multi-variable remainder has been stated we can continue with it's bounding properties. The absolute remainder of the null-th order Taylor series can be bounded by using the mean value theorem:

$$\begin{aligned} |R_0| &= |f(\mathbf{x}) - f(\mathbf{a})| \\ &\leq \left\| \int_0^1 (Df(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})\mathrm{d}t \right\| \\ &\leq \int_0^1 \|Df(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))\| \ \|\mathbf{x}-\mathbf{a}\| \, \mathrm{d}t \\ &\leq M\,\|\mathbf{x}-\mathbf{a}\| \end{aligned} \tag{18}$$

When using a higher order Taylor series the accompanying remainder is bounded by an inequality which makes use of a supremum norm [6]:

$$|R_k(\mathbf{x})| \leq \frac{1}{\beta!}\sup_{0\leq t\leq 1}\left\| D^\beta f(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})^\beta \right\| \tag{19}$$

For the case $f(\mathbf{x})$ is a vector function $\mathbf{f}(\mathbf{x}) : \mathbf{R}^n \to \mathbf{R}^m$, Eq. 19 can be extended to:

$$\|\mathbf{R}_k(\mathbf{x})\| \leq \frac{1}{\beta!} \sup_{0\leq t\leq 1} \left\| D^{\beta}\mathbf{f}(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})^{\beta} \right\| \tag{20}$$

This is the general case, in most cases related to aerospace applications, a continues function with continues derivatives is considered on a closed interval. This implies that the supremum norm can be replaced by a maximum norm:

$$\|\mathbf{R}_k(\mathbf{x})\|_{\infty} \leq \frac{1}{\beta!} \max_{0\leq t\leq 1} \left\| D^{\beta}\mathbf{f}(\mathbf{a}+t(\mathbf{x}-\mathbf{a}))(\mathbf{x}-\mathbf{a})^{\beta} \right\| \tag{21}$$

3 Incremental Non-linear Dynamic Inversion

The goal of an (I)NDI controller is to linearize the non-linear system dynamics. The closed loop transfer function of this linearizion than becomes equal to a pure integrator around which one can easily design a linear controller.

3.1 INDI derivation

The dynamical system under consideration is described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x},\mathbf{u}) \tag{22}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) \tag{23}$$

in which $\mathbf{f}$, $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^n$, $\mathbf{h} : \mathbf{R}^n \to \mathbf{R}^{2n}$, $\mathbf{x}$, $\mathbf{u} \in \mathbf{R}^n$ and $\mathbf{y} \in \mathbf{R}^{2n}$. For the inversion to be valid it is required that $\mathbf{g}(\mathbf{x},\mathbf{u})$ is continuously differentiable up until the first degree and $D\mathbf{g}$ should be invertible.

There are two ways one can derive an INDI controller [2] [4]. In this work the Taylor series method is used. The first step is to derive the first order Taylor series of Eq. (22):

$$\begin{aligned}\mathbf{T}(\mathbf{x},\mathbf{u},\mathbf{x}_0,\mathbf{u}_0) = \mathbf{f}(\mathbf{x}_0) &+ \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{x}-\mathbf{x}_0) \\ &+ \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{x}-\mathbf{x}_0) \\ &+ \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{u}-\mathbf{u}_0)\end{aligned} \tag{24}$$

Next $\mathbf{T}$ and $\mathbf{f}$ are replaced by the appropriate values of $\dot{\mathbf{x}}$:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{x}-\mathbf{x}_0)$$

$$+ \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{x}-\mathbf{x}_0)$$

$$+ \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} (\mathbf{u}-\mathbf{u}_0) \tag{25}$$

It is now presumed that the change in the state vector $\Delta\mathbf{x} = \mathbf{x}-\mathbf{x}_0$ is so small that we may neglect it. The change in input signal $\Delta\mathbf{u} = \mathbf{u}-\mathbf{u}_0$ is not neglected. The actual approximation is thus a first order Taylor Series expansion along $\mathbf{u}$ and a null-th order expansion along $\mathbf{x}$:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta\mathbf{u} \tag{26}$$

When solving for $\Delta\mathbf{u}$ the final INDI equation is obtained:

$$\Delta\mathbf{u} = \left(\left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} \right)^{-1} (\nu - \dot{\mathbf{x}}_0) \tag{27}$$

In Eq. 27 we have replace $\dot{\mathbf{x}}$ with the so called virtual control input ν. The value of the virtual control input can be obtained from a linear controller:

$$\nu = K_{\mathrm{p}}(\mathbf{x}_{\mathrm{r}} - \mathbf{x}) \tag{28}$$

in which K_{p} is the proportional gain, $\mathbf{x}_{\mathrm{r}}$ the set point or reference and $\mathbf{x}_{\mathrm{m}}$ the measured value of $\mathbf{x}$.

With the obtained value for $\Delta\mathbf{u}$ we can increment $\mathbf{u}_0$ to obtain the new controller output

$$\mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u} \tag{29}$$

hence the name Incremental NDI.

3.2 *INDI with model uncertainties*

In [3] a prove is given showing that INDI is robust towards model uncertainties. We will now state a summary of their derivations so that our later derivation of the INDI with a Taylor Series remainder is more clear.

The model described in Eq. (22) is only the model as we know it. In reality certain properties of the system may be unknown or uncertain. These properties can be modeled with the terms $\Delta\mathbf{f}(\mathbf{x}_0)$ and $\Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0)$, in which the subscript $\mathbf{u}$ indicates the partial derivative along $\mathbf{u}$.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}_0) + \Delta\mathbf{f}(\mathbf{x}_0) + G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0)\Delta\mathbf{u} + \Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0)\Delta\mathbf{u} \tag{30}$$

Note that the term $\Delta G_{\mathbf{x}}(\mathbf{x}_0, \mathbf{u}_0)\Delta \mathbf{x}$ has been left out because it is still assumed that $\Delta \mathbf{x}$ can be neglected.

The terms $\mathbf{f}(\mathbf{x}_0) + \Delta \mathbf{f}(\mathbf{x}_0)$ in Eq. (30) can be replaced by $\dot{\mathbf{x}}_0$ when it is assumed that $\dot{\mathbf{x}}_0$ is accurately measurable:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)\Delta \mathbf{u} + \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)\Delta \mathbf{u} \tag{31}$$

leaving only $\Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)$ as an unknown. Replacing $\Delta \mathbf{u}$ with the INDI controller from Eq. (27) yields:

$$\begin{aligned}
\dot{\mathbf{x}} &= \dot{\mathbf{x}}_0 + [G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) + \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0)] \left(\left. \frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{u})}{\partial \mathbf{u}} \right|_{\mathbf{x}=\mathbf{x}_0,\, \mathbf{u}=\mathbf{u}_0} \right)^{-1} (\nu - \dot{\mathbf{x}}_0) \\
&= \dot{\mathbf{x}}_0 + (G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) + \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0))\, G_{\mathbf{u}}^{-1}(\mathbf{x}_0, \mathbf{u}_0)(\nu - \dot{\mathbf{x}}_0) \\
&= \dot{\mathbf{x}}_0 + \left(I + \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0, \mathbf{u}_0)\right)(\nu - \dot{\mathbf{x}}_0) \\
&= -\Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0, \mathbf{u}_0)\dot{\mathbf{x}}_0 + \left(I + \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0, \mathbf{u}_0)\right) \nu \\
&= -B\dot{\mathbf{x}}_0 + (I + B)\nu
\end{aligned} \tag{32}$$

To simplify the equation the notation $B = \Delta G_{\mathbf{u}}(\mathbf{x}_0, \mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0, \mathbf{u}_0)$ has been introduced.

Continuing with Eq. (32), reference [3] obtains the closed loop transfer function Eq. (33) which includes the linear controller from Eq. (28). The closed loop is depicted in Fig. 1.

$$H_{\mathrm{cl}} = \frac{K_{\mathrm{p}}}{s + K_{\mathrm{p}}} \tag{33}$$

Because no uncertain terms show in the transfer function, [3] concluded that the INDI algorithm is robust towards model uncertainties.

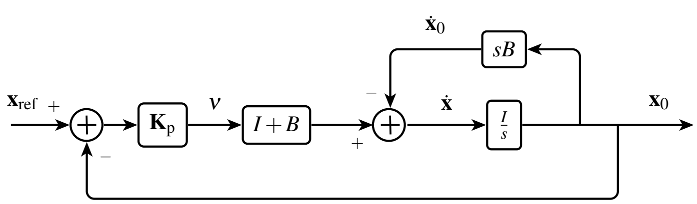

Fig. 1 Closed loop block diagram of the INDI controller and a system with uncertainties.

3.3 INDI under the influence of the Taylor series remainder

In the earlier section the INDI algorithm was derived without including the Taylor series remainder. In the following the remainder will be included to study the effects it has on the closed loop stability of the control loop. In an actual implementation it will not be included taking into account the lessons learned from the following derivation. The remainder could be included but this would render the advantage of the INDI algorithm, extensive model knowledge is not required, useless.

Including the remainder in the linearization of the model used for deriving the INDI algorithm, Eq. (26), results in:

$$\dot{\mathbf{x}} = \dot{\mathbf{x}}_0 + \left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0} \Delta\mathbf{u} + \mathbf{R}(\mathbf{x},\mathbf{u}) \tag{34}$$

As the INDI algorithm is based on a zeroth expansion along **x** and a first order along **u**, the remainder term depends on the first derivative of **x** and up until the second derivative of **u**. Also note that the actual remainder term as derived in the previous sections can be both positive and negative. It is only the absolute value which is bounded.

When continuing with the derivation of an INDI, Eq. (34) is solved for $\Delta\mathbf{u}$ which results in:

$$\Delta\mathbf{u} = \left(\left.\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{\mathbf{x}=\mathbf{x}_0,\,\mathbf{u}=\mathbf{u}_0} \right)^{-1} (\nu - \dot{\mathbf{x}}_0 - \mathbf{R}(\mathbf{x},\mathbf{u})) \tag{35}$$

This equation of the INDI algorithm is now entered in to the equations of motion with model uncertainties, Eq. (30):

$$\begin{aligned}\dot{\mathbf{x}} &= \dot{\mathbf{x}}_0 + (G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0) + \Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0))\, G_{\mathbf{u}}^{-1}(\mathbf{x}_0,\mathbf{u}_0)(\nu - \dot{\mathbf{x}}_0 - \mathbf{R}(\mathbf{x},\mathbf{u}))\\
&= \dot{\mathbf{x}}_0 + \left(I + \Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0,\mathbf{u}_0)\right)(\nu - \dot{\mathbf{x}}_0 - \mathbf{R}(\mathbf{x},\mathbf{u}))\\
&= -\Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0,\mathbf{u}_0)\dot{\mathbf{x}}_0 + \left(I + \Delta G_{\mathbf{u}}(\mathbf{x}_0,\mathbf{u}_0) G_{\mathbf{u}}^{-1}(\mathbf{x}_0,\mathbf{u}_0)\right)(\nu - \mathbf{R}(\mathbf{x},\mathbf{u}))\\
&= -B\dot{\mathbf{x}}_0 + (I+B)(\nu - \mathbf{R}(\mathbf{x},\mathbf{u}))\end{aligned} \tag{36}$$

The equation derived in Eq. (36) has a closed loop transfer function with the remainder as input equal to:

$$H_{\mathbf{R}} = \frac{-1}{s + K_{\mathrm{p}}} \tag{37}$$

Figure 2 displays the control loop. The remainder can thus be considered a disturbance which is rejected for values of $K_{\mathrm{p}} > 1$.

The influence of the remainder on the closed loop system is influenced in two ways. First the absolute value of the remainder is bounded by the system properties and the step size of the Taylor series. It is thus possible to reduce $|\mathbf{R}|$ by reducing the absolute step size. In the case of the INDI algorithm the step size is equal to $\omega_{\mathrm{r}} - \omega_0$ or in other words the error signal. The closer the measurement is to the set point the smaller the remainder. It is thus better to slowly vary the set point as

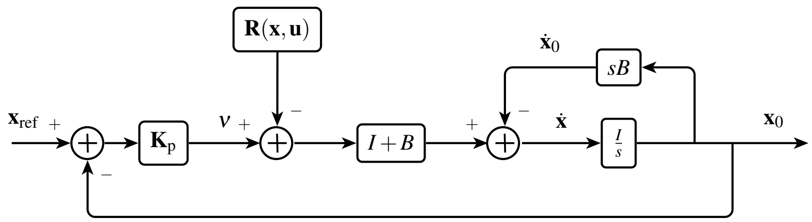

Fig. 2 Closed loop block diagram of the INDI controller including.

suppose to placing a step input on the system. The second possibility to decrease the influence of the remainder is to increase the gain K_p. The lower limit of K_p, $K_p > 1$ is determined by the ability of Eq. (3.3) to reject **R**. The upper limit is determined by the system, actuator dynamics, etc.

4 Example

To show the use of this theory an example follows. The choice has been made to use the model of an aerospace vehicle which controls its attitude by means of aerodynamic surfaces. Using this model the INDI algorithm for this particular model will be derived followed by a discussion on the stability of the system.

4.1 Model

In the model a number of variables are used: the state vector **x** which holds the rotational speed of the vehicle along it's three body axes, the input vector **u** holding the control surface deflections of the aileron, elevator and rudder, and last the measurement vector **y** which holds both the rotational speed and acceleration.

$$\mathbf{x} = \omega = [p\ q\ r]^T \tag{38}$$

$$\mathbf{u} = \delta = [\delta_a\ \delta_e\ \delta_r]^T \tag{39}$$

$$\mathbf{y} = [\omega\ \dot{\omega}]^T = [p\ q\ r\ \dot{p}\ \dot{q}\ \dot{r}]^T \tag{40}$$

As one can imagine this is a highly simplified model. This was chosen to focus on the effects different controller setting have on the bound of the remainder term.

A basic set of non linear equations where chosen which are influenced only by the rotational rate and control surface deflections:

$$\dot{\omega} = \mathrm{J}^{-1}\left(\mathbf{M}_a - \omega \times \mathrm{J}\omega\right) \tag{41}$$

in this equation the inertia is given by:

$$\mathrm{J} = \begin{bmatrix} \mathrm{J}_{xx} & 0 & 0 \\ 0 & \mathrm{J}_{yy} & 0 \\ 0 & 0 & \mathrm{J}_{zz} \end{bmatrix} \tag{42}$$

and the aerodynamic moment by $\mathbf{M}_\mathrm{a}$. The aerodynamic moment has three components: a part depending on ω, $\mathbf{M}_\mathrm{a}(\omega)$, a part depending on δ, $\mathbf{M}_\mathrm{a}(\delta)$, and a part which does not depend on neither, the general vector term $\mathbf{M}_\mathrm{a}$ will be used for this component from now on. The derivatives of interest along ω and δ are given by:

$$\begin{aligned} \mathbf{M}_{\mathrm{a}\omega} &= \frac{\partial \mathbf{M}_\mathrm{a}(\omega)}{\partial \omega} \\ &= \frac{1}{2}\rho V^2 S \begin{bmatrix} bC_{l_p} & 0 & bC_{l_r} \\ 0 & \bar{c}C_{m_q} & 0 \\ bC_{n_p} & 0 & bC_{n_r} \end{bmatrix} \end{aligned} \tag{43}$$

$$\begin{aligned} \mathbf{M}_{\mathrm{a}\delta} &= \frac{\partial \mathbf{M}_\mathrm{a}(\delta)}{\partial \delta} \\ &= \frac{1}{2}\rho V^2 S \begin{bmatrix} bC_{l_{\delta_\mathrm{a}}} & 0 & bC_{l_{\delta_\mathrm{r}}} \\ 0 & \bar{c}C_{m_{\delta_\mathrm{e}}} & 0 \\ bC_{n_{\delta_\mathrm{a}}} & 0 & bC_{n_{\delta_\mathrm{r}}} \end{bmatrix} \end{aligned} \tag{44}$$

In which the non-dimensional coefficients C are given along the moments l, m and n along the three body axes. The subscripts p, q and r indicate the rotational rates along the body axes and δ_a, δ_e and δ_r the control surface deflections of the aileron, elevator and rudder. The non-dimensional coefficients are made dimensional by multiplying them with a half times the air density ρ times the velocity squared V^2 times the wing area S. Depending on the axes the coefficient is multiplied by either the mean aerodynamic cord $\bar{c}$ or the wing span b. For this example it is presumed that all parameters are known exactly, [1] and [3] deal with parameter uncertainties.

This the given model at hand we now look back at Eq. (22) we can establish that the functions $\mathbf{f}(\omega)$ and $\mathbf{g}(\delta)$ are equal to:

$$\mathbf{f}(\omega) = \mathrm{J}^{-1}(\mathbf{M}_\mathrm{a} + \mathbf{M}_\mathrm{a}(\omega) - \omega \times \mathrm{J}\omega) \tag{45}$$

$$\mathbf{g}(\delta) = \mathrm{J}^{-1}\mathbf{M}_\mathrm{a}(\delta) \tag{46}$$

Based on the model Eqs. (46) and (46), the controller implementation of Eqs. (27), (28) and (29) now becomes becomes:

$$\nu = K_\mathrm{p}(\omega_\mathrm{r} - \omega_\mathrm{m}) \tag{47}$$

$$\begin{aligned} \Delta\delta &= \left(\mathrm{J}^{-1}\mathbf{M}_{\mathrm{a}\delta}\right)^{-1}(\nu - \dot{\omega}_\mathrm{m}) \\ &= \mathbf{M}_{\mathrm{a}\delta}^{-1}\mathrm{J}(\nu - \dot{\omega}_\mathrm{m}) \end{aligned} \tag{48}$$

$$\delta = \delta_\mathrm{m} + \Delta\delta \tag{49}$$

In which ω_{r} is the set point and ω_{m} the measured value of the rotational rate. Also the measured or estimated value of the rotational acceleration, $\dot{\omega}_{\mathrm{m}}$, is required.

4.2 Taylor series remainder

As stated in section 3.3 the remainder term created by deriving an INDI controller is comprised out of the first order derivatives along ω and the second order derivative along δ. The variables as used in the multi-variable Taylor series remainder, Eq. (21), have in this example the following implementation for the part that belongs to the zeroth order Taylor series along ω:

$$\mathbf{a} = \omega_{\mathrm{m}} = [p_{\mathrm{m}}\ q_{\mathrm{m}}\ r_{\mathrm{m}}]^T \tag{50}$$
$$\mathbf{x} = \omega_{\mathrm{r}} = [p_{\mathrm{r}}\ q_{\mathrm{r}}\ r_{\mathrm{r}}]^T \tag{51}$$

and in the case of the first order Taylor series along δ:

$$\mathbf{a} = \delta_{\mathrm{m}} = \left[\dot{\delta}_{\mathrm{am}}\ \dot{\delta}_{\mathrm{em}}\ \dot{\delta}_{\mathrm{rm}}\right]^T \tag{52}$$
$$\mathbf{x} = \delta = \left[\dot{\delta}_{\mathrm{ar}}\ \dot{\delta}_{\mathrm{er}}\ \dot{\delta}_{\mathrm{rr}}\right]^T \tag{53}$$

Do note that in this example it is assumed that there is no measurement delay. In [3] a prediction algorithm is proposed which is designed to overcome problems arising from this delay.

Continuing with the remainder terms, the zeroth order remainder requires the following derivative:

$$\begin{aligned} D^1\mathbf{f}(\omega) &= \frac{\partial \mathbf{f}(\omega)}{\partial \omega} \\ &= \frac{\partial}{\partial \omega}\left(\mathrm{J}^{-1}\left(\mathbf{M}_{\mathrm{a}} + \mathbf{M}_{\mathrm{a}}(\omega) - \omega \times \mathrm{J}\omega\right)\right) \\ &= \mathrm{J}^{-1}\left(\mathbf{M}_{\mathrm{a}\omega} - \left[\frac{\partial}{\partial p}(\omega \times \mathrm{J}\omega)\ \frac{\partial}{\partial q}(\omega \times \mathrm{J}\omega)\ \frac{\partial}{\partial r}(\omega \times \mathrm{J}\omega)\right]\right) \end{aligned} \tag{54}$$

The coefficients in this equation are all known and used in common aerodynamic models. This is not the case with the first order remainder:

$$\begin{aligned} D^{(1,0)}\mathbf{g}(\delta) &= \frac{\partial \mathbf{g}(\delta)}{\partial \delta} \\ &= \frac{\partial}{\partial \delta}\mathrm{J}^{-1}\mathbf{M}_{\mathrm{a}}(\delta) \\ &= \mathrm{J}^{-1}\mathbf{M}_{\mathrm{a}\delta} \\ D^{(2,0)}\mathbf{g}(\delta) &= \frac{\partial^2 \mathbf{g}(\delta)}{\partial \delta^2} \end{aligned} \tag{55}$$

$$= \frac{\partial^2}{\partial \delta^2} \mathrm{J}^{-1} \mathbf{M}_\mathrm{a}(\delta)$$
$$= \mathrm{J}^{-1} \mathbf{M}_{\mathrm{a}\delta^2} \tag{56}$$
$$D^{(1,1)} \mathbf{g}(\delta) = \frac{\partial^2 \mathbf{g}(\delta)}{\partial \delta \partial \omega}$$
$$= \frac{\partial^2}{\partial \delta \partial \omega} \mathrm{J}^{-1} \mathbf{M}_\mathrm{a}(\delta)$$
$$= \mathrm{J}^{-1} \frac{\partial}{\partial \omega} \mathbf{M}_{\mathrm{a}\delta} \tag{57}$$

Second order derivatives and derivatives along ω of the control parameters are not commonly used. In the following results it was assumed that the zeroth order remainder is dominate. This assumption is based on the fact that for short term motions, linear models capture the dynamics well. In these model second order control parameter derivatives are not used and are thus presumed of lesser importance.

This results in a remainder term which is bounded by:

$$|\mathbf{R}_0(\omega)| = \left\| \mathrm{J}^{-1} \left(\mathbf{M}_{\mathrm{a}\omega} - \left[\frac{\partial}{\partial p} (\omega \times \mathrm{J}\omega) \ \frac{\partial}{\partial q} (\omega \times \mathrm{J}\omega) \ \frac{\partial}{\partial r} (\omega \times \mathrm{J}\omega) \right] \right) (\omega_\mathrm{r} - \omega_\mathrm{m}) \right\|_\infty \tag{58}$$

4.3 Discussion of the simulation results

Using numerical software the model has been simulated. During the simulation two parameters have been varied: the controller gain and the set point function. The controller gain has been set equal to $K_\mathrm{p} = 1$ and $K_\mathrm{p} = 10$. For the set point variation two option where chosen: a step and a ramp, both end at the same values and vary both the p and q desired values. The effects these variations have on the model output are depicted in the Figs. 3 till 9. Figure 3 depicts the bounding value of the remainder during the different cases. Figures 4 till 6 depict the rotational rates of the vehicle, Figs. 7 till 9 depict the control surface deflections. As indicated in section 3.3 the bounding value of the remainder is influenced by these two parameters, this is clearly visible from the results.

Stated in section 3.3 is that the transfer function with the remainder as input becomes unstable for controller gains $K_\mathrm{p} \leq 1$. This was not found in the results, a considerable decrease of the bounding value is however visible when increasing the controller gain. It is suspected that this is partially due to the fact that the set point is followed much better as can be seen in Figs. 4 till 6. This was the second point stated in section 3.3 which would influence the remainder limit. Using a set point which stays closer to the current value clearly decreases the limit considerable. The more slowy varying ramp function clearly has a much lower value.

5 Conclusion

From the analysis of the INDI closed loops with the Taylor series remainder it has become clear that the remainder is a disturbance which can be rejected under certain conditions. The absolute remainder will decrease for a small error signal and large gains. The addition of the remainder does not effect the robustness of the algorithm with regard to model uncertainties.

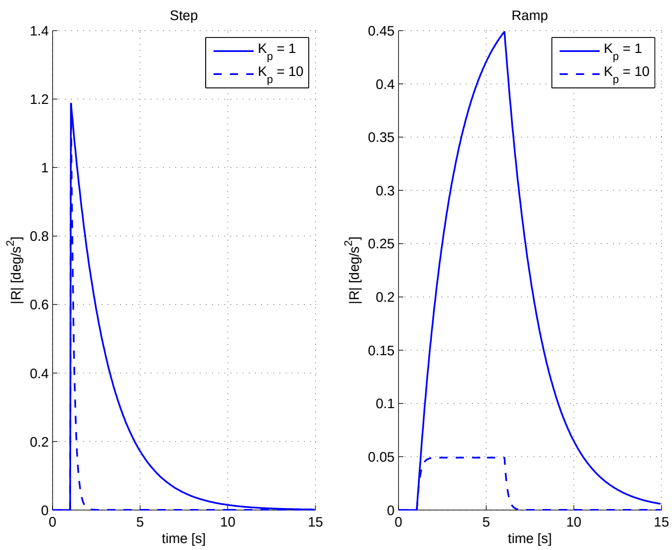

Fig. 3 Maximum values of the remainder for different gains and set point variations.

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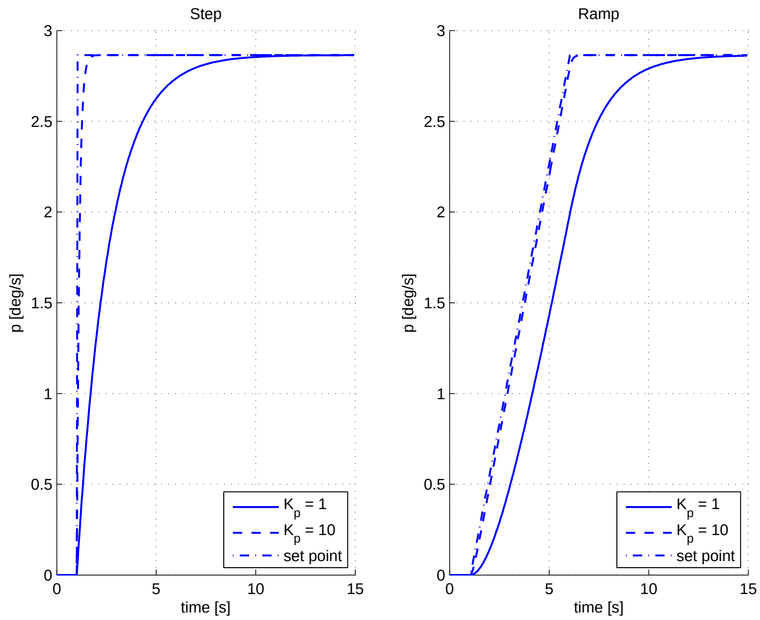

Fig. 4 The response of the angular rate along the x axis for different gains and set point variations.

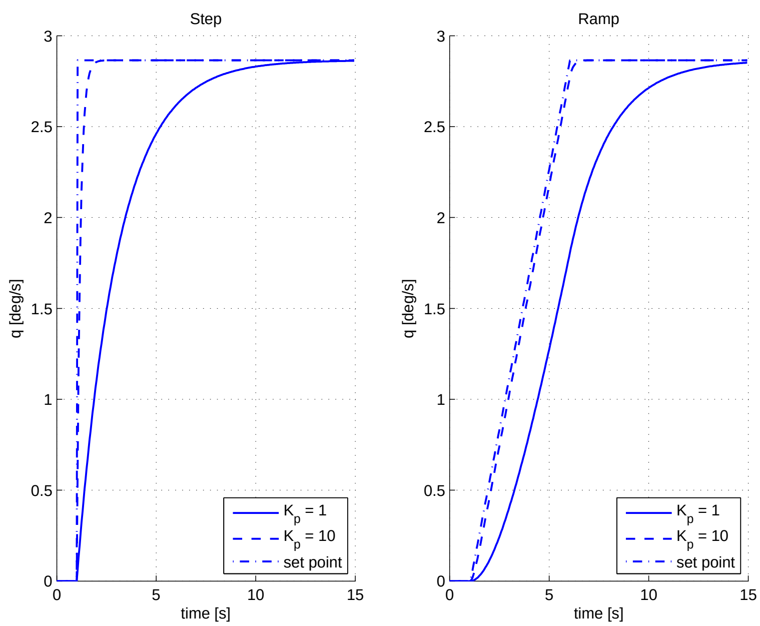

Fig. 5 The response of the angular rate along the y axis for different gains and set point variations.

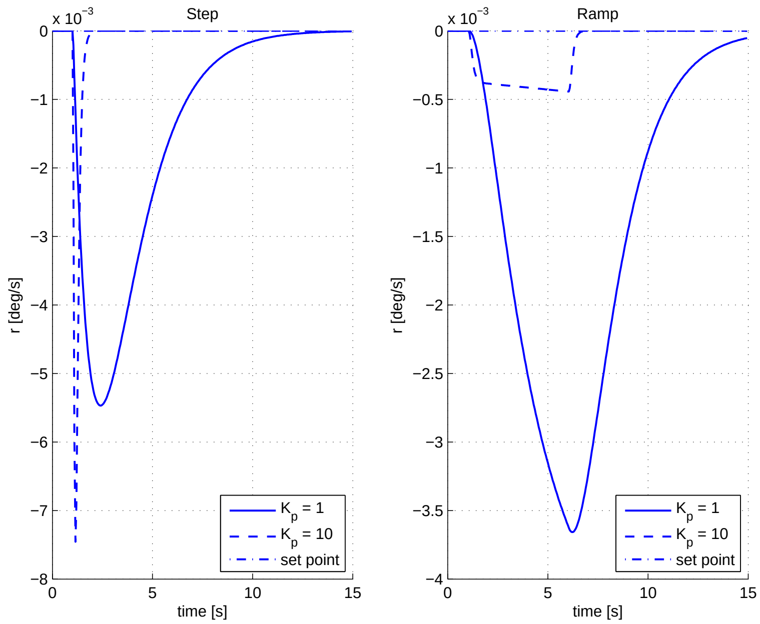

Fig. 6 The response of the angular rate along the z axis for different gains and set point variations.

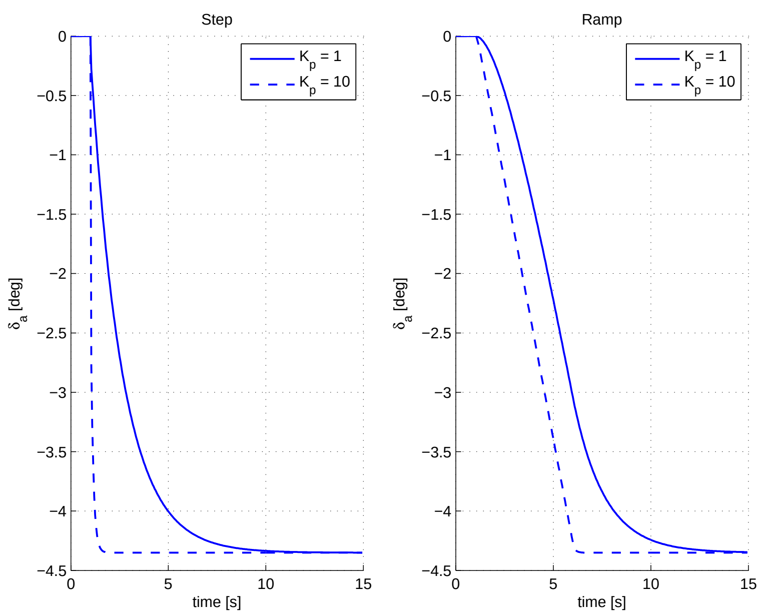

Fig. 7 Aileron deflections for different gains and set point variations.

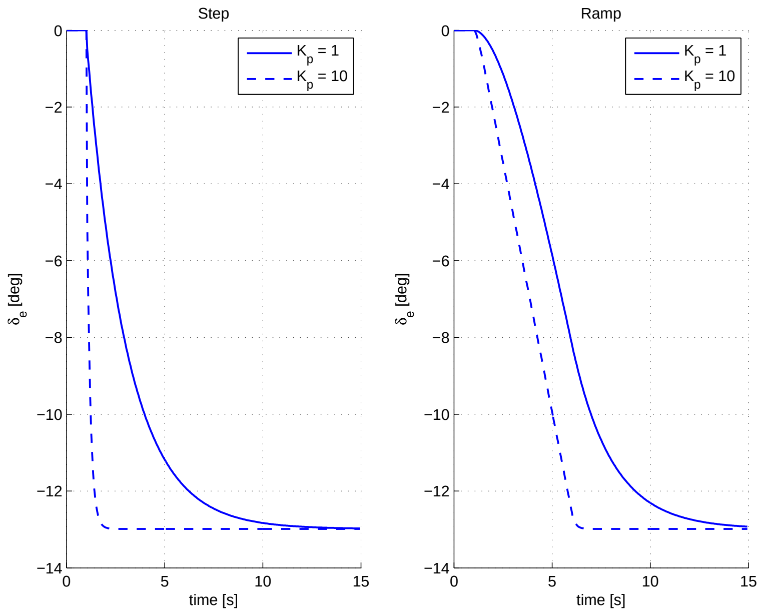

Fig. 8 Elevator deflections for different gains and set point variations.

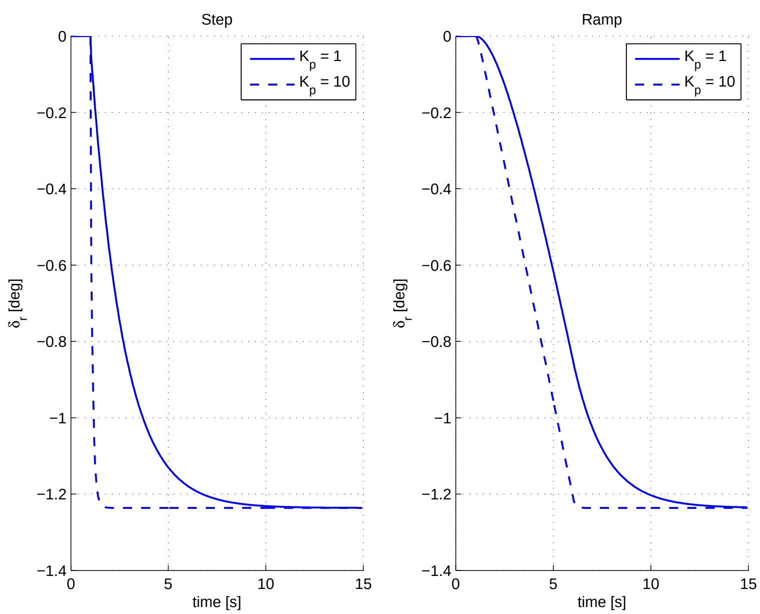

Fig. 9 Rudder deflections for different gains and set point variations.