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Model Formulation of Pursuit Problem with Two Pursuers and One Evader

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Abstract

We study a model differential zero-sum game, which can be regarded as an idealized variant of the final stage of a space pursuit, in which two pursuing objects and one evader are involved. Results of numeric constructions of level sets of the value function for qualitatively different cases of the game parameters and results of simulation of optimal motions are presented.

1 Introduction and Problem Formulation

1) In the paper, a model differential zero-sum game with two pursuers and one evader is studied. Three inertial objects moves in the straight line. The dynamics descriptions for pursuers P_1 and P_2 are

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$$\begin{aligned} &\ddot{z}_{P_1} = a_{P_1}, && \ddot{z}_{P_2} = a_{P_2}, \\ &\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ &|u_1| \le \mu_1, && |u_2| \le \mu_2, \\ &a_{P_1}(t_0) = 0, && a_{P_2}(t_0) = 0. \end{aligned} \tag{1}$$

Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers; a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\ddot{z}_E = a_E, \quad \dot{a}_E = (v - a_E)/l_E, \quad |v| \le \nu, \quad a_E(t_0) = 0. \tag{2}$$

Let us fix some instants T_1 and T_2. At the instant T_1, the miss of the first pursuer with respect to the evader is computed, and at the instant T_2, the miss of the second one is calculated:

$$r_{P_1,E}(T_1) = |z_E(T_1) - z_{P_1}(T_1)|, \quad r_{P_2,E}(T_2) = |z_E(T_2) - z_{P_2}(T_2)|. \tag{3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is regarded as the *second player*. The resultant miss is computed by the following formula:

$$\varphi = \min\{r_{P_1,E}(T_1),\ r_{P_2,E}(T_2)\}. \tag{4}$$

At any instant t, both players know exact values of all state coordinates z_{P_1}, $\dot{z}_{P_1}$, a_{P_1}, z_{P_2}, $\dot{z}_{P_2}$, a_{P_2}, z_E, $\dot{z}_E$, a_E. The vector composed of these components is denoted by z. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game. One needs to construct the value function $(t, z) \mapsto \mathcal{V}(t, z)$ of this game and optimal (or quasioptimal) strategies of the players.

2) Up to now, there are a lot of publications dealing with differential games where one group of objects pursues another group; concerning games with linear dynamics see, for example, works [1,4,6,11,12]. The problem under consideration has two pursuers and one evader. So, from the point of view of number of objects, it is the simplest one. On the other hand, strict mathematical studies of problems "group-on-group" usually include quite strong assumptions onto the dynamics of objects, dimension of the state vector, and conditions of termination. Unlike, this paper considers the problem without any assumptions of these types.

3) Let us describe a practical problem, whose reasonable simplification gives the model game (1)–(4). Suppose that two pursuing objects attack the evading one with high velocities. They can be rockets or aircrafts in the horizontal

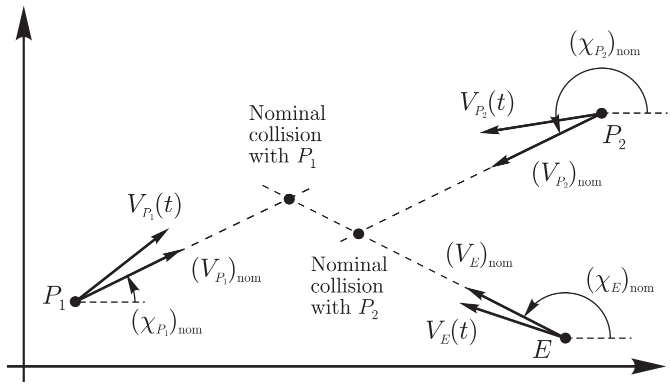

Fig. 1 Scheme of the nominal motions in the pursuit problem with weak-maneuvering objects

plane (Fig. 1). A nominal motion of the first pursuer is chosen such that at the instant T_1 the exact capture occurs. In the same way, a nominal motion of the second pursuer is chosen (the capture is at the instant T_2). But indeed, the real positions of the objects differ from the nominal ones. Moreover, the evader using its control can change its trajectory but not essentially, without sharp turns. Coordinated efforts of the pursuers are computed during the process by the feedback method to minimize the resultant miss, which is the minimum of the distances at the instants T_1 and T_2 from the first and second pursuers, respectively, to the evader.

Assume that we can choose a line (in Fig. 1, it is a horizontal line) such that the major components of velocities of all three objects are directed along it. Then, the misses at the instants T_1 and T_2, can be computed along a direction orthogonal to such a line ignoring difference of positions along this line.

The passage from the original non-linear dynamics to a dynamics, which is linearized with respect to the nominal motions, gives [13, 14] the problem under consideration.

2 Passage to Two-Dimensional Differential Game

At first, let us pass to the relative geometric coordinates

$$y_1 = z_E - z_{P_1}, \quad y_2 = z_E - z_{P_2} \tag{5}$$

in dynamics (1), (2), and payoff function (4). After this, we have the following notations:

$$\begin{aligned}
&\ddot{y}_1 = a_E - a_{P_1}, && \ddot{y}_2 = a_E - a_{P_2}, \\
&\dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\
&\dot{a}_E = (v - a_E)/l_{P_1}, && |u_2| \le \mu_2, \\
&|u_1| \le \mu_1,\ |v| \le \nu, && \varphi = \min\{|y_1(T_1)|,\ |y_2(T_2)|\}.
\end{aligned} \tag{6}$$

State variables of system (6) are y_1, $\dot{y}_1$, a_{P_1}, y_2, $\dot{y}_2$, a_{P_2}, a_E; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinate y_1 at the instant T_1 and on the coordinate y_2 at the instant T_2.

A standard approach to study linear differential games with fixed terminal instant and payoff function depending on some target coordinates of the state vector at the terminal instant is to pass to new state coordinates (see, for example, [7, 8]) that can be treated as values of the target coordinates forecasted to the terminal instant under zero controls. Often, these coordinates are called the *zero effort miss coordinates* [13, 14]. In our case, we have two instants T_1 and T_2, but coordinates computed at these instants are independent; namely, at the instant T_1, we should take into account $y_1(T_1)$ only, and at the instant T_2, we use the value $y_2(T_2)$. This fact allows us to use the mentioned approach when solving the differential game (6). With that, we pass to new state coordinates x_1 and x_2, where $x_1(t)$ is the value of y_1 forecasted to the instant T_1 and $x_2(t)$ is the value of y_2 forecasted to the instant T_2.

The forecasted values are computed by formula

$$x_i = y_i + \dot{y}_i\tau_i - a_{P_i} l_{P_i}^2 h(\tau_i/l_{P_i}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \tag{7}$$

Here, x_i, y_i, $\dot{y}_i$, a_{P_i}, and a_E depend on t; $\tau_i = T_i - t$. Function h is described by the relation $h(\alpha) = e^{-\alpha} + \alpha - 1$. Emphasize that the values τ_1 and τ_2 are connected to each other by the relation $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$. It is very important that $x_i(T_i) = y_i(T_i)$. Let $X(t, z)$ be a two-dimensional vector composed of the variables x_1, x_2 defined by formulae (5), (7).

The dynamics in the new coordinates x_1, x_2 is the following [9]:

$$\begin{aligned}
&\dot{x}_1 = -l_{P_1} h(\tau_1/l_{P_1}) u_1 + l_E h(\tau_1/l_E) v, && |u_1| \le \mu_1,\ |u_2| \le \mu_2, \\
&\dot{x}_2 = -l_{P_2} h(\tau_2/l_{P_2}) u_2 + l_E h(\tau_2/l_E) v, && |v| \le \nu.
\end{aligned} \tag{8}$$

The payoff function is $\varphi\big(x_1(T_1), x_2(T_2)\big) = \min\{|x_1(T_1)|,\ |x_2(T_2)|\}$.

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ.

Note that the control u_1 (u_2) affects only the horizontal (vertical) component $\dot{x}_1$ ($\dot{x}_2$) of the velocity vector $\dot{x} = (\dot{x}_1, \dot{x}_2)^{\mathrm{T}}$. When $T_1 = T_2$, the second summand in dynamics (8) is the same for $\dot{x}_1$ and $\dot{x}_2$. Thus, the component of the velocity vector $\dot{x}$ depending on the second player control is directed at any instant t along the bisectrix of the first and third quadrants of the plane

x_1, x_2. When $v = +\nu$, the angle between the axis x_1 and the velocity vector of the second player is 45°; when $v = -\nu$, the angle is 225°. This property simplifies the dynamics in comparison with the case $T_1 \neq T_2$.

Let $x = (x_1, x_2)^{\mathrm{T}}$ and $V(t, x)$ be the value of the value function of game (8) at the position (t, x). From general results of the differential game theory, it follows that $\mathcal{V}(t, z) = V\big(t, X(t, z)\big)$. This relation allows one to compute the value function of the original game (1)–(4) using the value function for game (8).

For any $c \geq 0$, a level set (a Lebesgue set) $W_c = \big\{(t, x) : V(t, x) \leq c\big\}$ of the value function in game (8) can be treated as the solvability set for the considered game with the result not greater than c, that is, for a differential game with dynamics (8) and the terminal set

$$M_c = \big\{(t, x) : t = T_1,\ |x_1| \leq c\big\} \bigcup \big\{(t, x) : t = T_2,\ |x_2| \leq c\big\}.$$

When $c = 0$, one has the situation of the exact capture. The exact capture means equality to zero, at least, one of $x_1(T_1)$ and $x_2(T_2)$. Let $W_c(t) = \{x : (t, x) \in W_c\}$ be the time section (t-section) of the set W_c at the instant t. Similarly, let $M_c(t)$ for $t = T_1$ and $t = T_2$ be the t-section of the set M_c at the instant t.

Comparing dynamics capabilities of each of pursuers P_1 and P_2 and the evader E, one can introduce the parameters [9, 14] $\eta_i = \mu_i/\nu$, $\varepsilon_i = l_E/l_{P_i}$, $i = 1, 2$. They define the shape of the solvability sets in the individual games P_1–E and P_2–E. Namely, depending on values of η_i and $\eta_i\varepsilon_i$ (which are not equal to 1 simultaneously), there are 4 cases [14] of the solvability set evolution (see Fig. 2):

- expansion in the backward time (a strong pursuer);
- contraction in the backward time (a weak pursuer);
- expansion until some backward time instant and further contraction;
- contraction until some backward time instant and further expansion (if the solvability set still has not broken).

Respectively, given combinations of pursuers' capabilities in individual games and durations T_1, T_2 (equal/different), there are significant number of variants for the problem with two pursuers and one evader.

The ideology of solving the game used by us is the following. Choose the parameters η_i, ε_i, and, also, the instants T_i, $i = 1, 2$; then, using some fine grid for values of c, we compute the level sets W_c of the value function. After that, we can build optimal or quasioptimal strategies of the first and second players.

Nowadays, different workgroups suggested many algorithms for numeric solution of differential games of quite general type (see, for example, [2, 3, 5, 10, 15]). Problem (8) has the second order on the state variable and can be rewritten as

$$\dot{x} = \mathcal{D}_1(t)u_1 + \mathcal{D}_2(t)u_2 + \mathcal{E}(t)v, \qquad |u_1| \leq \mu_1,\ |u_2| \leq \mu_2,\ |v| \leq \nu. \tag{9}$$

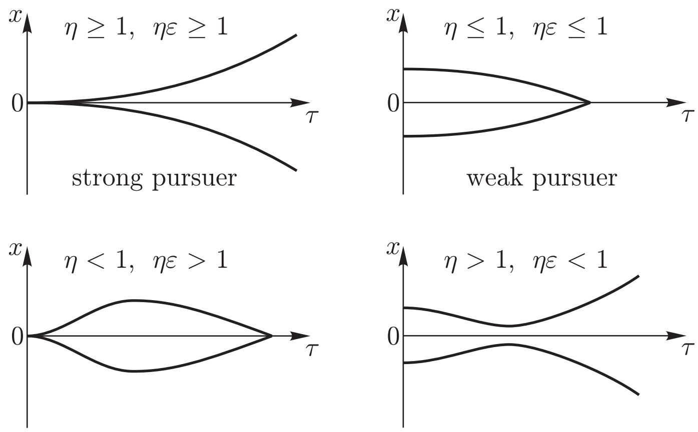

Fig. 2 Variants of the solvability set evolution in an individual game

Here, $x = (x_1, x_2)^{\mathrm{T}}$; vectors $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$, and $\mathcal{E}(t)$ look like

$$\mathcal{D}_1(t) = \big(-l_{P_1} h((T_1 - t)/l_{P_1})^{\mathrm{T}}, 0\big), \qquad \mathcal{D}_2(t) = \big(0, -l_{P_2} h((T_2 - t)/l_{P_2})\big)^{\mathrm{T}},$$
$$\mathcal{E}(t) = \big(l_E h((T_1 - t)/l_E), \, l_E h((T_2 - t)/l_E)\big)^{\mathrm{T}}.$$

The control of the first player has two independent components u_1 and u_2. The vector $\mathcal{D}_1(t)$ ($\mathcal{D}_2(t)$) is directed along the horizontal (vertical) axis. The second player's control v is scalar. When $T_1 = T_2$, the angle between the axis x_1 and the vector $\mathcal{E}(t)$ equals 45°; when $T_1 \neq T_2$, the angle changes in time.

Due to peculiarity of our problem, we use special methods for constructing level sets of the value function.

3 Maximal Stable Bridge: Control with Discrimination

A level set W_c of the value function V is a maximal stable bridge (MSB) breaking on the terminal set M_c [7,8].

Let $T_1 = T_2$. Denote $T_f = T_1$. Using the concept of MSB from [7,8], we can say that W_c is the set maximal by inclusion in the space $t \leq T_f$, x such that $W_c(T_f) = M_c(T_f)$ and the *stability* property holds: for any position $(t_*, x_*) \in W_c(t_*)$, $t_* < T_f$, any instant $t^* > t_*$, $t^* \leq T_f$, any constant control v of the second player, which obeys the constraint $|v| \leq \nu$, there is a measurable control $t \to \big(u_1(t), u_2(t)\big)$ of the first player, $t \in [t_*, t^*)$, $|u_1(t)| \leq \mu_1$, $|u_2(t)| \leq \mu_2$, guiding system (8) from the state x_* to the set $W_c(t^*)$ at the instant t^*.

The stability property assumes a discrimination of the second player by the first one: the choice of the first player's control in the interval $[t_*, t^*)$ is made after the second player announces his control in this interval.

It is known (see [7, 8]) that any MSB is close. The set $W'_c(t) = \mathrm{cl}\big(R^2 \setminus W_c(t)\big)$ (the symbol cl denotes the operation of closure) is the time section of MSB W'_c for the second player at the instant t. The bridge terminates at the instant T_f on the set $M'_c(T_f) = \mathrm{cl}\big(R^2 \setminus M_c(T_f)\big)$. If the initial position of system (8) is in W'_c and if the first player is discriminated by the second one, then the second player is able to guide the motion to the set $M'_c(T_f)$ at the instant T_f. Thus, $\partial W_c = \partial W'_c$. It is proved that for any initial position $(t_0, x_0) \in \partial W_c$, the value c is the best guaranteed result for the first (second) player in the class of feedback controls.

Due to symmetry of dynamics (8) and the set $W_c(T_f)$ with respect to the origin, one gets that for any $t \le T_f$ the time section $W_c(t)$ is symmetric also.

If $T_1 \ne T_2$, then there is no any appreciable complication in constructing MSBs for the problem considered in this paper in comparison with the case $T_1 = T_2$. Indeed, let $T_1 > T_2$. Then in the interval $(T_2, T_1]$ in (8), we take into account only the dynamics of the variable x_1 when building the bridge W_c backwardly from the instant T_1. With that, the terminal set at the instant T_1 is taken as $M_c(T_1) = \{(x_1, x_2) : |x_1| \le c\}$. When the constructions are made up to the instant T_2, we add the set $M_c(T_2)$, that is, we take

$$W_c(T_2) = W_c(T_2 + 0) \bigcup \big\{(x_1, x_2) : |x_2| \le c\big\},$$

and further constructions are made on the basis of this set.

So, our tool for finding a level set of the value function in game (8) corresponding to a number c is the backward procedure for constructing a MSB with the terminal set M_c. Presence of an idealized element (the discrimination of the opponent) allowed us to create effective numeric methods for backward construction of MSBs.

The solvability set with the index equal to c in the individual game $P1$–E ($P2$–E) is MSB built in the coordinates t, x_1 (t, x_2) and terminating at the instant T_1 (T_2) on the set $|x_1| \le c$ ($|x_2| \le c$). Its t-section, if it is non-empty, is a segment in the axis x_1 (x_2) symmetric with respect to the origin. In the plane x_1, x_2, this segment corresponds to a vertical (horizontal) strip of the same width near the axis x_2 (x_1). It is evident that when $t \le T_1$ ($t \le T_2$), such a strip is contained in the section $W_c(t)$ of MSB W_c of game (8) with the terminal set M_c.

4 Results of Numeric Constructions of Maximal Stable Bridges

Case of strong pursuers. In the case of two strong pursuers, the t-sections of MSBs in individual games $P1$–E and $P2$–E grow with increasing of the backward time. This gives that for any $c \geq 0$ and any $t \leq \bar{t} = \min\{T_1, T_2\}$ the set $W_c(t)$ includes a cross near the axes x_1, x_2, which expands with decreasing t.

Let us give results of constructing t-sections $W_c(t)$ for the following values of the game parameters: $\mu_1 = 2$, $\mu_2 = 3$, $\nu = 1$, $l_{P_1} = 1/2$, $l_{P_2} = 1/0.857$, $l_E = 1$.

Equal terminal instants. Let $T_1 = T_2 = 6$. Fig. 3 shows results of constructing the set W_0 (that is, with $c = 0$). In the figure, one can see several time sections $W_0(t)$ of this set. The bridge has a quite simple structure. At the initial instant $\tau = 0$ of the backward time (when $t = 6$), its section coincides with the target set, which is the union of two coordinate axes. Further, at the instants $t = 4$, 2, 0, the cross thickens, and two triangles are added to it. The widths of the vertical and horizontal parts of the cross correspond to sizes of MSBs in the individual games with the first and second pursuers. These triangles are located in the II and IV quadrants (where the signs of x_1 and x_2 are different, in other words, when the evader is between the pursuers). They give the zone where the exact capture is possible only under collective actions of both pursuers.

Time sections $W_c(t)$ of other bridges W_c, $c > 0$, have a shape similar to $W_0(t)$.

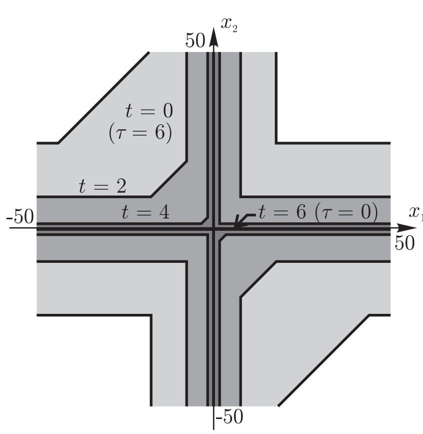

Fig. 3 Two strong pursuers, equal terminal instants: time sections of the maximal stable bridge W_0

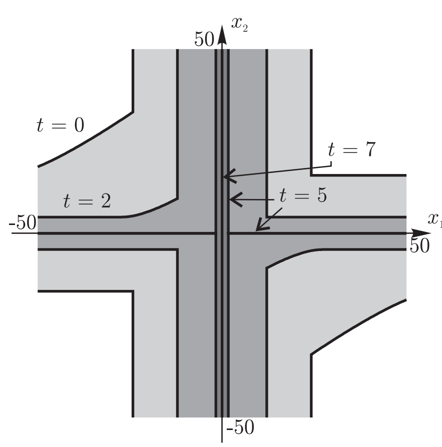

Fig. 4 Two strong pursuers, different terminal instants: time sections of the maximal stable bridge W_0

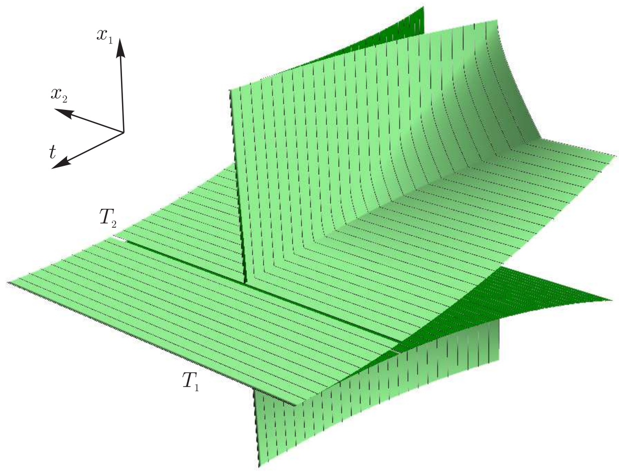

Fig. 5 Strong pursuers, different terminal instants: 3D-view of the set W_0

Different terminal instants. Let $T_1 = 7$, $T_2 = 5$. Results of constructing the set W_0 are given in Fig. 4. When $t < 5$, time sections $W_0(t)$ grow both horizontally and vertically; two additional triangles appear, but in this case they are curvilinear. In Fig. 5, the set W_0 is shown in the three-dimensional space t, x_1, x_2.

The given results are typical for the case of strong pursuers. When $T_1 = T_2$, the sets $W_c(t)$ can be described analytically. This was done in paper [9]. Also, there the case $T_1 \neq T_2$ was studied. But for it, only an upper approximation of the sets $W_c(t)$ was obtained.

Case of weak pursuers. Since in the case of weak pursuers the t-sections of MSBs in individual games $P1$–E and $P2$–E contract with growth of the backward time and become empty at some instant, the set $W_c(t)$ for any $c \geq 0$ with decreasing of t loses infinite sizes along axes x_1 and x_2.

The most surprising fact discovered during the numeric study was that the connected set $W_c(t)$ with decreasing of t loses connectedness and disjoins into two separate parts.

Take the parameters $\mu_1 = 0.9$, $\mu_2 = 0.8$, $\nu = 1$, $l_{P_1} = l_{P_2} = 1/0.7$, $l_E = 1$. Let us show results for the case of different terminal instants only: $T_1 = 9$, $T_2 = 7$. Since in this variant the evader is more maneuverable than the pursuers, the first player cannot guarantee the exact capture.

The set W_c in the space t, x_1, x_2 for $c = 2.0$ is shown in Fig. 6. During evolution of the sections $W_{2.0}(t)$ in t, they change their structure at some instants. These places are marked by drops in the constructed surface of the set.

One strong and one weak pursuers. Let us take the following parameters: $\mu_1 = 2$, $\mu_2 = 1$, $\nu = 1$, $l_{P_1} = 1/2$, $l_{P_2} = 1/0.3$, $l_E = 1$. Now the

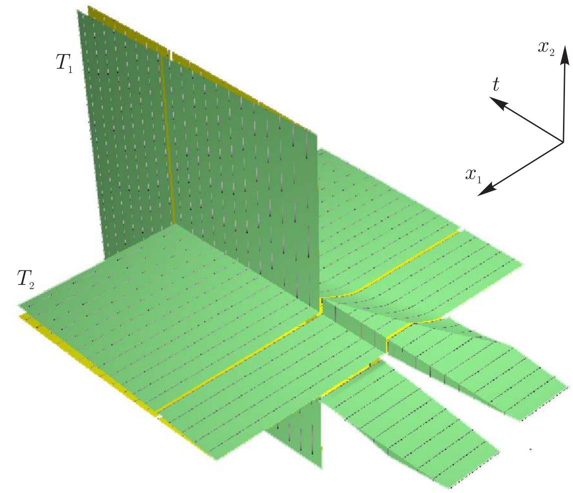

Fig. 6 Two weak pursuers, different terminal instants: 3D-view of the set $W_{2.0}$

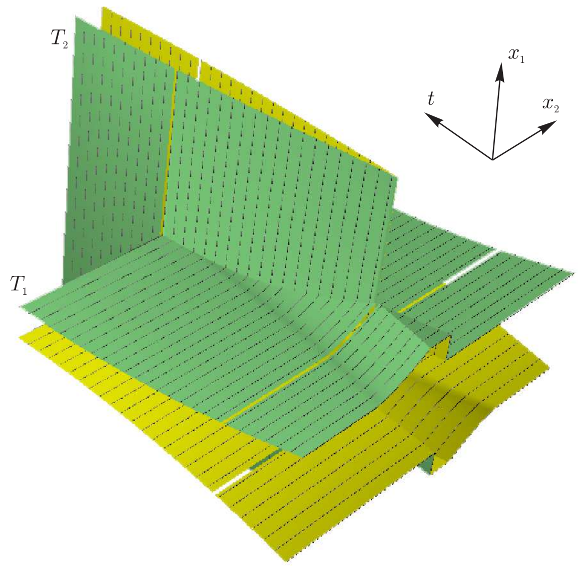

Fig. 7 One strong and one weak pursuers, different termination instants: 3D-view of the set $W_{5.0}$

evader is more maneuverable than the second pursuer, and an exact capture by this pursuer is unavailable. Assume $T_1 = 5$, $T_2 = 7$.

In Fig. 7, a three-dimensional view of MSB $W_{5.0}$ is shown. The part along the axis x_1 of its time section $W_{5.0}(t)$ contracts with decreasing of τ, and breaks further. The part along the axis x_2 grows. After breaking the indi-

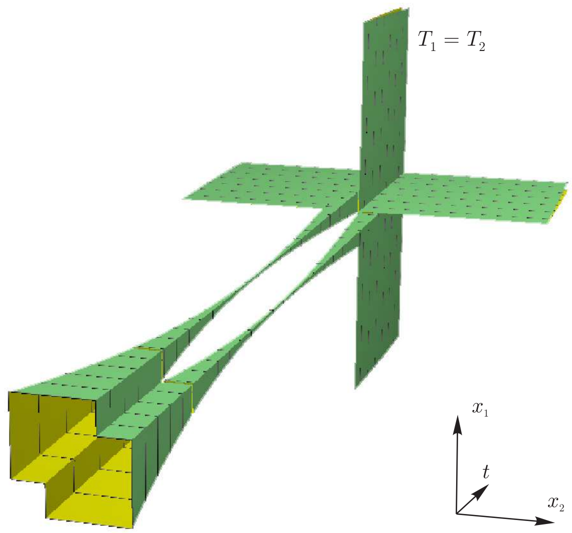

Fig. 8 Varying advantage of the pursuers, equal termination instants: 3D-view of the maximal stable bridge $W_{1.315}$

vidual MSB $P2$–E (and respective collapse of the part of the cross along the axis x_1), there is a strip along the axis x_2 only with two additional parts determined by the joint actions of both pursuers.

Varying advantage of pursuers. Consider a variant when both pursuers P_1 and P_2 are equal, with that at the beginning of the backward time, the bridges in the individual games contract and further expand. Choose the game parameters in such a way that for some c the section $W_c(t)$ of MSB W_c with decreasing of t disjoins into two parts, which join back with further decreasing of t.

Parameters of the game are $\mu_1 = \mu_2 = 1.5$, $\nu = 1$, $l_{P_1} = l_{P_2} = 1/0.25$, $l_E = 1$. Termination instants are equal: $T_1 = T_2 = 15$.

A three-dimensional view of MSB $W_{1.315}$ is shown in Fig. 8.

5 Control on the Basis of Switching Lines

A control based on the switching lines assumes separation of the state space x_1, x_2 to some cells at instants from some grid in time. In each cell, every scalar control keeps some extreme value. The time grid should contain intants, when a player chooses its control in a discrete scheme. Under a discrete control scheme [7, 8] with the step Δ, a control chosen at the instant t_s is kept until the instant $t_{s+1} = t_s + \Delta$. At the position $\big(t_{s+1}, x(t_{s+1})\big)$, a new control value is chosen, etc.

1) In the game under consideration, the first player has two scalar controls u_1, u_2, which are bounded by the inequalities $|u_1| \leq \mu_1$, $|u_2| \leq \mu_2$. The component of the velocity of system (9), which is affected by the control u_1, is connected to the vector $\mathcal{D}_1(t)$ and is horizontal in our case. The component corresponding to the control u_2 is connected to the vector $\mathcal{D}_2(t)$ and is directed vertically.

To separate the plane x_1, x_2 into parts, in which the control u_1 takes one of the extreme values $u_1 = +\mu_1$ or $u_1 = -\mu_1$, we study the change of the value function at the instant t in lines parallel to the vector $\mathcal{D}_1(t)$, that is, in horizontal lines.

In the problem that we investigate, the following property is true (except situations of varying advantage of the pursuers) for each horizontal line. The restriction of the value function $V(t, \cdot)$ to a horizontal line is a function having only one interval of local minimum, which is either a point, or a segment, or the entire line. With that, the restriction grows when the argument goes from the interval of minimum.

Considering an arbitrary horizontal line, we can gather the points of minimum of the restriction of the value function to this line. We take an arbitrary point from such an interval of minimum as a point for the switching line of the control u_1. Taking points from all horizontal lines in such a way, we obtain a switching line $\Pi_1(t)$ separating the plane x_1, x_2 into two parts. In the part, where the vector $\mathcal{D}_1(t)$ is directed from the switching line, we define the control u_1^* equal to $-\mu_1$, and in the another part, it is equal to $+\mu_1$. During numeric constructions, the switching line $\Pi_1(t)$ is built on the basis of some number (quite great, but finite) of time sections $W_{c_j}(t)$ of the level sets of the value function for some collection $\{c_j\}$ of values of the parameter c.

In the same way using corresponding objects, the switching line $\Pi_2(t)$ can be built for the control u_2.

The control of the first player based on the switching lines $\Pi_1(t)$ and $\Pi_2(t)$, we call *quasioptimal* because we assume that in the switching lines, the control u_1 (u_2) is taken arbitrary from the interval $[-\mu_1, +\mu_1]$ ($[-\mu_2, +\mu_2]$). For the cases of "strong" and "weak" pursuers, it can be proved that such a choice is optimal indeed. But for the case of varying advantage of the pursuers, it is possible that for some small neighborhood of the switching lines we need some additional information about the value function. The authors have not studied this question yet.

Fig. 9 shows the typical picture of the time sections $W_c(t)$ of the level sets and switching lines $\Pi_1(t)$ and $\Pi_2(t)$ for the case of varying advantage of the pursuers.

Emphasize once more that the switching lines depend on time t, and the choice of the control is defined by the current state position of the system with respect to the corresponding switching line. The vectors $\mathcal{D}_1(t)$ and $\mathcal{D}_2(t)$ are used. Drawing a ray from the point $x(t)$ with the directing vector $\mathcal{D}_i(t)$, one can decide whether it crosses the switching line $\Pi_i(t)$. If it does not, then $u_i^*\big(t, x(t)\big) = -\mu_i$, if it crosses, then $u_i^*\big(t, x(t)\big) = +\mu_i$.

Thus, to organize computations of the discrete control scheme of the first player, we should keep in memory of the computer a collection of the switching lines in some time grid.

2) The direction of the action of the second player's scalar control v is defined by the vector $\mathcal{E}(t)$. Its direction is constant in the case $T_1 = T_2$ and changes in time if $T_1 \neq T_2$. When constructing the switching lines for the second player, we analyze points of local maxima and minima of restrictions of the value function to lines parallel to the vector $\mathcal{E}(t)$. For each of these lines, the collection of all points of minima and maxima can consists, generally speaking, of several intervals. Nevertheless, their number is small. This allows us to take corresponding points from them and to constitute some lines, which separate the plane x_1, x_2 into parts, in which the control v keeps one of its extreme values $-\nu$ or $+\nu$.

To construct $v^*\big(t, x(t)\big)$, we use the vector $\mathcal{E}(t)$. Compute how many times (even or odd) a ray with the beginning at the point $x(t)$ and the directing vector $\mathcal{E}(t)$ crosses the second player switching lines. If the number of crosses is even (absence of crosses means that the number equals zero and is even), then we take $v^*\big(t, x(t)\big) = +\nu$; otherwise, $v^*\big(t, x(t)\big) = -\nu$.

The typical picture of the switching lines of the second player is given in Fig. 10 for the case of varying advantage of the pursuers. Here, one can see 6 domains of constancy of the second player's control v. Direction of

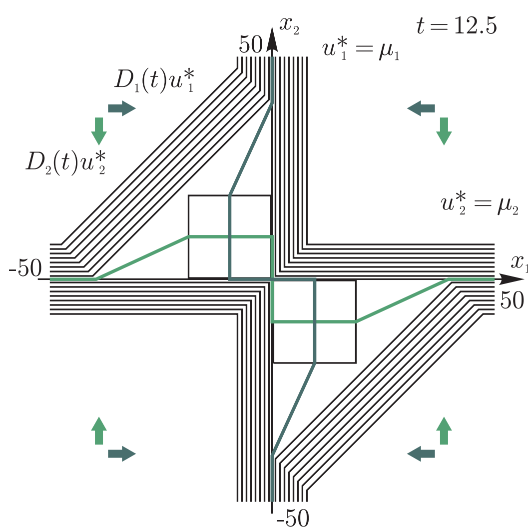

Fig. 9 The case of varying advantage of the pursuers. The typical picture of the switching lines for the first player; the dark green line is for the control u_1, the light green one is for the control u_2

its action are shown by arrows. In the lines, which are composed of points of local maxima of the value function, the control can be taken arbitrary from the interval $[-\nu, +\nu]$. But in the lines consisting of the point of local minima, from the theoretic point of view, only extreme values $-\nu$ and $+\nu$ are allowed, which push the system from the switching line. Due to errors of numeric construction of the swiching lines, this way of control can lead to a motion in a sliding regime along the switching line (that changes in time). Such a motion can be unoptimal from the point of view of the second player. Assuming this situation to be almost impossible, we regard the suggested method of the second player's control as a quasioptimal one.

6 Optimal Motion Simulation Results

Let the pursuers P_1, P_2, and the evader E move in the plane. This plane is called the *original geometric space*. At the initial instant t_0, velocities of all objects are parallel to the horizontal axis and sufficiently larger than the possible changes of the lateral velocity components. The components of object velocities, which are parallel to the horizontal axis, are constant. Magnitudes of these components are such that the rendezvous of the objects P_1 and E happens at the instant T_1, and the objects P_2 and E encounter at the instant T_2. The dynamics of lateral motion is described by relations (1), (2); the resultant miss is given by formula (4).

The initial lateral velocities and accelerations are assumed to be zero:

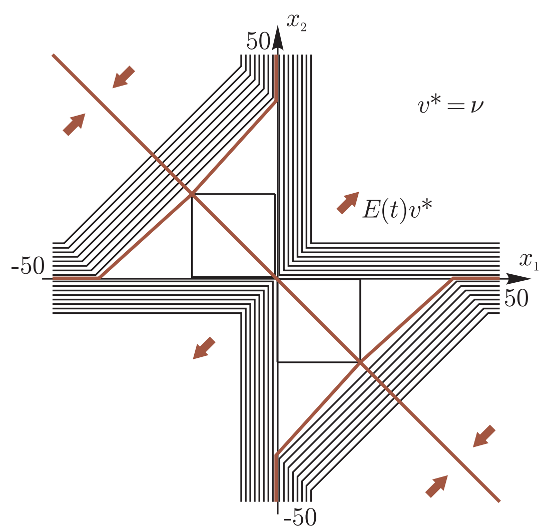

Fig. 10 The case of varying advantage of the pursuers. The typical picture of the switching lines for the second player for the same instant $t = 12.5$ as in Fig. 9

$$\dot{z}^0_{P_1} = \dot{z}^0_{P_2} = \dot{z}^0_E = 0, \quad a^0_{P_1} = a^0_{P_2} = a^0_E = 0.$$

The simulation is made for the following parametes of the game:

$$\mu_1 = \mu_2 = 1.1,\ \nu = 1,\ l_{P_1} = l_{P_2} = 1/0.6,\ l_E = 1,\ T_1 = T_2 = 20.$$

The parameters are such that the pursuers can achieve a higher acceleration than the evader, but they are more inertial, that is, the achievement of the extreme acceleration lasts longer than the evader's one. We have

$$\eta_i = \mu_i/\nu = 1.1 > 1,\ \eta_i\varepsilon_i = \eta_i \cdot \frac{l_E}{l_{P_i}} = 1.1 \cdot 0.6 = 0.66 < 1,\ i = 1,\ 2.$$

So, we consider the case of varying advantage of the pursuers. In this situaton, the exact capture is not guaranteed.

In Figs. 11 and 12, the horizontal axis is denoted by the symbol d. The coordinate d shows the longitudinal position of the objects. Controls of the objects affect the vertical (lateral) coordinate.

Fig. 11 shows the optimal trajectories of the objects for the following initial positions at the instant $t_0 = 0$:

$$z^0_{P_1} = -130,\ z^0_{P_2} = 100,\ z^0_E = 0.$$

The initial deviations are so large that the second pursuer (the upper one) is unable reach the evader, even applying its extremal control. But the first pursuer (the lower one) has a quite small miss, which, nevertheless, is still non-zero.

In Fig. 12, the optimal trajectories are given for the initial positions

$$z^0_{P_1} = -20,\ z^0_{P_2} = 10,\ z^0_E = 0.$$

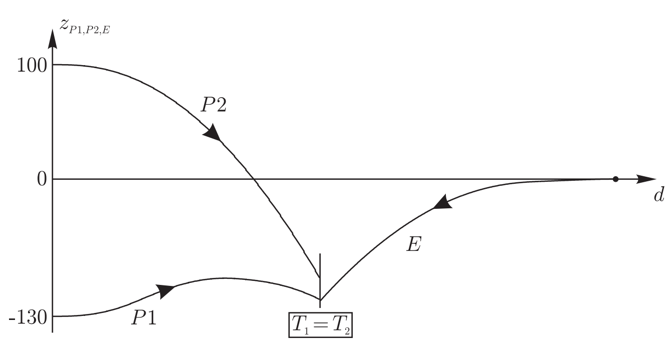

Fig. 11 Optimal trajectories in the case of varying advantage of the pursuers; large initial deviations

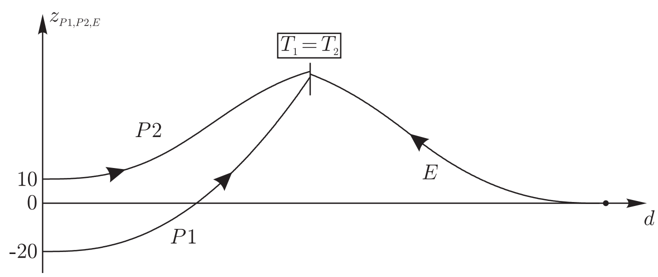

Fig. 12 Optimal trajectories in the case of varying advantage of the pursuers; small initial deviations

Now, both pursuers have small terminal misses, but they are non-zero due to the advantage of the evader at the final stage of the pursuit. Note that the evader is just in the middle between the pursuers at the instant $T_1 = T_2$: such a position provides the maximal possible payoff value for him.

7 Conclusion

For a model zero-sum differential game with two pursuing ond one evading objects, a numeric solution is obtained: the level sets of the value function, quasioptimal strategies on the basis of switching lines, simulation of motions using the suggested strategies. A complete investigation of the problem can be made because the original formulation allows an equivalent presentation with two-dimensional state vector in the plane of coordinates of one-dimensional forecasted misses (zero-effort miss coordinates). Similar problems are much harder if the miss between each pursuer and the evader are computed in a two-dimensional geometric space.

Acknowledgements

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