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Fault-tolerant Spacecraft Magnetic Attitude Control

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Abstract This work is concerned with the development of a suboptimal control algorithm for Markovian jump-linear systems, and its application to fault-tolerant spacecraft magnetic attitude control. For completeness, the jump-linear quadratic optimal controller with full state and mode information is presented. Relaxing the assumption of perfect mode information, a similar optimal control problem is formulated where the mode is observed via discrete measurements. The elements of the measurement matrix, i.e. the probabilities for correct and wrong mode observations are assumed known. The optimal controller is developed, which requires an exponentially growing computational burden, and a suboptimal controller is proposed that only requires knowledge of the current mode measurement. This controller is finite memory and possess some of the classical linear quadratic regulator features such as the linear state feedback structure and a state quadratic optimal cost-to-go. The performances of the suggested algorithm are illustrated through extensive Monte-Carlo simulations on a simple numerical example. A realistic fault-tolerant spacecraft magnetic attitude controller is developed based on the proposed approach. The attitude controller succeeds in mitigating the destabilizing effect of corrupted mode observations while being computationally efficient.

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1 Introduction

Jumping parameters processes, also called step processes, are commonly used in order to model abrupt parametric variations in the models of dynamical systems due for instance to sensors or actuators failures. Such processes also allow to incorporate state discontinuities in the differential equations that govern the system dynamics and they are useful when approximating non-linear systems by a set of linearized models [1]. Research focused on the development of optimal control algorithms when both the state of the system and the value of the step process ("the mode") are instantaneously available. Stochastic minimum principles were given for non-linear [2] and linear [3] continuous-time systems where the jumping process is modeled as a Markov step processes. Further works on quadratic regulation of Jump Linear systems investigated the influence of additive diffusive process noise on the optimal control [4], the case where the mode transition probability matrix is a stochastic variable [5], and where the state discontinuities have random amplitudes [6]. Following the stochastic Dynamic Programming approach, quadratic regulators of Jump Linear systems were developed in the continuous-time [7] and discrete-time [8] settings. The working assumptions in Ref. 7 were that full state and mode information was available and that a Markov feedback control law was sought. The jumping parameter was modeled as a finite-state continuous-time homogenous Markov step process with ν possible values and a known differential transition matrix. The highlights of that work are that the control is linear in the dynamic state, with gain parameters that are determined by the simultaneous backward propagation of a set of ν coupled Riccati differential equations, where the coupling captures the effect of the stochastic mode-switching on the state dynamics. These gains, however, depend on the current value of the mode. In applications, the mode variations would represent parametric variations in linear dynamics, and such variations are usually not directly accessible to physical measurements. For this reason, the model described is of limited practical interest. On the other hand, when the assumption of full information is relaxed, optimal control and estimation solutions are known to yield algorithms with exponentially growing memory [9]. This motivated the development of suboptimal but practical algorithms that can operate on systems with process and measurement noises and where the state only is observed. To achieve practicality such algorithms rely on approximations in the hypothesis pruning or in the expression for the optimal return function [10]. Such a sub-optimal controller is suggested in Ref. 11 under the assumptions of known continuous state while the mode is detected after some random delay, which probabilistic model is given.

This work is concerned with the development of an approximate jump-linear quadratic controller with partial mode information and its application to spacecraft attitude control. This work briefly reviews the general jump-linear quadratic controller under full state and mode information. Relaxing the assumption of perfect mode information, the mode is observed via discrete-valued measurements with known conditional probability distribution. The optimal solution is presented, the conflict between optimality and practicality is noted, and a computationally efficient suboptimal algorithm is suggested. The proposed algorithm has got some of the classical Linear Quadratic regulator features: recursion, linear state feedback, state quadratic optimal cost-to-go, which are direct consequences of the nested property of the information patterns and of the full continuous-state information. The gain computation, however, depends on the specific information structure. The proposed suboptimal controller is a finite memory controller and a look-up table for the gains can be computed off-line. Comparative results of an extensive Monte-Carlo simulation for a simple system illustrate the efficiency of the proposed algorithm that mitigates the destabilizing effect of corrupted mode observations. The detailed developments of the optimal and the proposed suboptimal jump-linear quadratic controllers are presented. The novel approximate controller is compared to other hybrid systems controller. The modeling of failures in the context of spacecraft attitude dynamics are described and their embedding in the framework of jump-linear system is addressed. Focus is made on failures in magnetic sensing and actuators, which are common on-board small satellites. Simulations for an Earth orbiting three-axis magnetically stabilized small spacecraft is performed under realistic modeling assumptions and results from extensive Monte-Carlo simulations are presented.

2 Jump-Linear Quadratic Control Problem

Consider the following discrete time jump-linear dynamical system

$$\mathbf{x}_{k+1} = A(y_k)\mathbf{x}_k + B(y_k)\mathbf{u}_k + \mathbf{w}_k \qquad k = 0, 1, \ldots, N-1 \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$, $\mathbf{u}_k \in \mathbb{R}^m$, $\mathbf{w}_k \in \mathbb{R}^n$, $A(y_k) \in \mathbb{R}^{n\times n}$, $B(y_k) \in \mathbb{R}^{n\times m}$, $\{\mathbf{w}_k :$ k=0,1,...,N-1$\}$ is a zero mean white sequence, with known covariance matrices, W_k, and $\{y_k : k = 0, 1, \ldots, N-1\}$ is a scalar autonomous Markov chain with finite state space $S_y = \{1, 2, \ldots, \nu\}$ and transition matrix P; that is,

$$P[i, j] = p_{ij} = Pr(y_{k+1} = j \mid y_k = i) \tag{2}$$

The sequences $\{y_k\}$, and $\{\mathbf{w}_k\}$ are assumed to be independent one from the other and from the initial conditions $\mathbf{x}_0$. The system represented by the joint state $\{\mathbf{x}_k, y_k\}$, which dynamics is governed by Eq. (1) and by the above assumption on $\{y_k\}$, belongs to the class of Markovian jump-linear systems [7]. Typically, the continuous states, $\mathbf{x}_k$, represent physical quantities while the discrete state, y_k, represents a logical state that describes a "mode" of operation of the system. In the context of fault modeling, occurrences of failures are modeled by the switch of the Markov chain state, y_k, from one value to another, according to a priori transition probabilities. Let J denote the following quadratic cost function

$$J = E\left\{\sum_{k=0}^{N-1} \|\mathbf{x}_{k+1}\|^2_{Q(y_k)} + \|\mathbf{u}_k\|^2_{R(y_k)}\right\} \tag{3}$$

where $Q(y_k) \in \mathbb{R}^{n\times n}$, $Q(y_k) \geq 0$, $R(y_k) \in \mathbb{R}^{m\times m}$, $R(y_k) > 0$, for $k = 0, 1, \ldots, N-1$. The jump-linear quadratic control problem (JLQ) [13, Ch. 2] consists in finding the sequence $\{\mathbf{u}_k\}$ that minimizes the cost J in Eq. (3) subject to Eq. (1) and under the assumptions of state and mode full information. The set of admissible controls is the set of functions of the available information. This type of controller is called "fault-tolerant" because its design simultaneously accounts for failure-driven dynamics of the physical states. The random variables $A(y_k)$, $B(y_k)$, $Q(y_k)$, and $R(y_k)$, will be denoted by A_k, B_k, Q_k, and R_k, respectively. The solution to the full information JLQ problem [2, 8] yields a linear controller where the gains are computed as follows:
Initialize the algorithm with

$$K\{y_N, N\} = 0 \qquad y_N = 1, 2, \ldots, \nu \tag{4}$$

For $k = N-1, N-2, ..., 0$, and $y_k = 1, 2, \ldots, \nu$, compute,

$$\tilde{K}\{y_k, k+1\} = \sum_{j=1}^{\nu} p_{y_k,j}\, K\{j, k+1\} + Q_k \tag{5}$$

$$M\{y_k, k\} = \left(R_k + B_k{}^T \tilde{K}\{y_k, k+1\} B_k\right)^{-1} B_k^T \tilde{K}\{y_k, k+1\} A_k \tag{6}$$

$$K\{y_k, k\} = A_k{}^T \tilde{K}\{y_k, k+1\} (A_k - B_k M\{y_k, k\}) \tag{7}$$

Notice that Eqs. (5)- (7) are functions of the mode y_k and consist, thus, of a set of ν coupled backward Riccati-like equations. These computations yield ν sequences of gain matrices $M\{y_k, k\}$, $k = 0, 1, \ldots, N-1$, which are mode dependent and can be computed off-line. The control algorithm is then applied as follows: at each time step k, y_k and $\mathbf{x}_k$ are known and the adequate gain matrix $M\{y_k, k\}$ is selected. The control, $\mathbf{u}_k^*$, is

$$\mathbf{u}_k^* = -M_k\ \mathbf{x}_k \tag{8}$$

where M_k denotes M$\{y_k, k\}$, for simplicity. Notice that the linear structure of the control law Eq. (8) is an outcome of the optimal control derivation and not an assumption. As a by product, the optimal cost-to-go is computed as follows:

$$J_k^* = \|\mathbf{x}_k\|^2_{K_k} + \sum_{i=k}^{N-1} tr\left(W_i \tilde{K}_{i+1}\right) \tag{9}$$

where $K_k, \tilde{K}_{i+1}$ denote, respectively, the matrices $K\{y_k, k\}$ and $\tilde{K}\{y_k, _{k+1}\}$, from Eqs. (4), (7). The optimal cost-to-go J_k^* is computed forward along with the optimal control sequence.

3 JLQ Control with Discrete Mode Observations

3.1 Problem formulation

Consider the same discrete time jump-linear dynamical system as given in Eq. (1) with the same assumptions on the dynamics of the mode, on the noises, and on the state-$\mathbf{x}_k$ information. It is also assumed that the mode y_k is observed and that the mode measurements are discrete-valued random variables. Let $\{z_k : k = 0, 1, \ldots, N-1\}$ denote the sequence of observations of the mode y_k and let $S_z = \{1, 2, \ldots, \nu\}$ denote the finite state space of each observation z_k. The mode measurement model is characterized by a priori probabilities of correct and missed detections as follows

$$\Pi_k[i, j] \stackrel{\triangle}{=} \Pr(z_k = i \,|y_k = j) \tag{10}$$

Considering the following quadratic cost function,

$$J = E\left\{\sum_{k=0}^{N-1} \|\mathbf{x}_{k+1}\|^2_{Q(y_k)} + \|\mathbf{u}_k\|^2_{R(y_k)}\right\} \tag{11}$$

where $Q(y_k) \in \mathbb{R}^{n\times n}$, $Q(y_k) \geq 0$ and $R(y_k) \in \mathbb{R}^{m\times m}$, $R(y_k) > 0$, $k = 0, 1, \ldots, N-1$, a sequence of control vectors $\{\mathbf{u}_k\}$ is sought that minimizes J subject to Eq. (1), under the assumption of full information on $\mathbf{x}_k$. The set of admissible controls is the set of functions of the present and past history of the state $\mathbf{x}_k$ and of the observations z_k.

3.2 Optimal and suboptimal solution

The optimal solution to this problem was developed via Dynamic Programming (Appendix A) and is summarized next. The detailed developments will be provided in the final manuscript.
Initialize the algorithm with

$$\tilde{S}_N = 0 \tag{12}$$

For $k = N-1, N-2, ..., 0$, compute

$$S_k\left(\mathcal{X}^{k+1},\mathcal{Z}^{k+1},y_k\right)=\tilde{S}_{k+1}+Q_k \tag{13}$$

$$\Gamma_k\left(\mathcal{X}^{k},\mathcal{Z}^{k}\right)=\left(\overline{B_k^T S_k B_k+R_k}\right)^{-1}\overline{B_k^T S_k A_k} \tag{14}$$

$$\tilde{S}_k\left(\mathcal{X}^{k},\mathcal{Z}^{k}\right)=\overline{A_k^T S_k A_k}-\overline{A_k^T S_k B_k}\Gamma_k \tag{15}$$

where $\mathcal{X}^k \triangleq \{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_k\}$ and $\mathcal{Z}^k \triangleq \{z_0,z_1,\ldots,z_k\}$ denote the histories of $\{\mathbf{x}_k\}$ and $\{\mathbf{z}_k\}$, respectively. The variables $\overline{F}$ in Eqs. (14) and (15) are defined as follows:

$$\overline{F}(\mathcal{X}^k,\mathcal{Z}^k)\triangleq E\{F \mid \mathcal{X}^k,\mathcal{Z}^k\} \tag{16}$$

Considering the conditioning sequence in Eq. (16) the computation of these conditional expectations would require a growing memory size. Two approximate solutions are examined, first only the mode observation histories $\mathcal{Z}^k$ are retained, i.e., by computing the following conditional expectation

$$\tilde{\overline{F}}(\mathcal{Z}^k)=E\{F \mid \mathcal{Z}^k\}$$

The computations of $\tilde{\overline{F}}(\mathcal{Z}^k)$ are performed as follows (Proof in Appendix B):

$$\tilde{\overline{F}}(\mathcal{Z}^k)=\sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y}F\left(y_k,\mathcal{Z}^k,z_{k+1}\right)\sum_{y_{k+1}}^{\nu}\left\{\Pr\left(z_{k+1}|y_{k+1}\right)\Pr\left(y_{k+1}|y_k\right)\right\}$$
$$\times\frac{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_k|y_{k-1}\right)\Pr\left(y_{k-1},z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}}{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_{k-1},z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}} \tag{17}$$

which yields the following expressions as a function of the model parameters:

$$\tilde{\overline{F}}(\mathcal{Z}^{k-1},z_k=\zeta)=\sum_{i=1}^{\nu}\sum_{j=1}^{\nu}F(j,i,\mathcal{Z}^k) \tag{18}$$

$$\times\frac{\sum_{r=1}^{\nu}\left\{\pi_{\mathrm{ir}}^{(k+1)}p_{\mathrm{jr}}\right\}\sum_{l=1}^{\nu}\left\{p_{\mathrm{lj}}\Lambda_{\zeta\mathrm{l}}^{(k-1)}(\mathcal{Z}^{k-1})\Pr\left(y_{k-1}=l\right)\right\}}{\sum_{m=1}^{\nu}\left\{\Lambda_{\zeta\mathrm{m}}^{(k-1)}(\mathcal{Z}^{k-1})\Pr\left(y_{k-1}=m\right)\right\}} \tag{19}$$

where p_{ij} denotes the element at location i,j in the transition matrix P, $\pi_{\mathrm{ij}}^{(k)}$ denotes the element at location i,j in the matrix Π_k and $\Lambda_{\mathrm{ij}}^{(n)}(\mathcal{Z}^n)$ can be computed as follows:
Initialize the computation with

$$\Lambda_{\mathrm{ij}}^{(0)}(\zeta)=\frac{\sum_{r=1}^{\nu}\left\{\pi_{\mathrm{ir}}^{(1)}p_{\mathrm{jr}}\right\}\pi_{\zeta\mathrm{j}}^{(0)}\Pr\left(y_0=j\right)}{\sum_{\tau=1}^{\nu}\left\{\pi_{\zeta\tau}^{(0)}\Pr\left(y_0=\tau\right)\right\}} \tag{20}$$

For $k=1,2,...,n$, compute

$$\Lambda_{\mathrm{ij}}^{(k)}(\mathcal{Z}^{k-1}, z_k = \zeta) = \frac{\sum_{r=1}^{\nu}\left\{\pi_{\mathrm{ir}}^{(k+1)} p_{\mathrm{jr}}\right\}\sum_{l=1}^{\nu}\left\{p_{\mathrm{lj}}\Lambda_{\zeta\mathrm{l}}^{(k-1)}(\mathcal{Z}^{k-1})\Pr\left(y_{k-1}=l\right)\right\}}{\sum_{m=1}^{\nu}\left\{\Lambda_{\zeta\mathrm{m}}^{(k-1)}(\mathcal{Z}^{k-1})\Pr\left(y_{k-1}=m\right)\right\}} \tag{21}$$

The resulting algorithm for that approximation also has substantial computational burden of about ν^k, which leads to the second suggested approximate solution by retaining only the current mode observation z_k, i.e., by computing the following conditional expectation

$$\overline{\overline{F}}(z_k) = E\{F \mid z_k\}$$

The computations of $\overline{\overline{F}}(z_k)$ are performed as follows (Proof in Appendix B):

$$\overline{\overline{F}}\left(z_k\right) = \sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y} F\left(y_k, z_{k+1}\right) \times \frac{\Pr\left(z_k|y_k\right)\Pr\left(y_k\right)\sum_{y_{k+1}\in S_y}\left\{\Pr\left(z_{k+1}|y_{k+1}\right)\Pr\left(y_{k+1}|y_k\right)\right\}}{\sum_{y_k\in S_y}\left\{\Pr\left(z_k|y_k\right)\Pr\left(y_k\right)\right\}} \tag{22}$$

which yields the following expressions as a function of the model parameters:

$$\overline{\overline{F}}\left(\zeta\right) = \sum_{i=1}^{\nu}\sum_{j=1}^{\nu} F(j,i)\frac{\pi_{\zeta\mathrm{j}}^{(k)}\sum_{r=1}^{\nu}\left\{p_{\mathrm{rj}}^{(k)}\Pr\left(y_0=r\right)\right\}\sum_{l=1}^{\nu}\left\{\pi_{\mathrm{il}}^{(k+1)}p_{\mathrm{jl}}\right\}}{\sum_{m=1}^{\nu}\left\{\pi_{\zeta\mathrm{m}}^{(k)}\sum_{\mu=1}^{\nu}\left\{p_{\mu\mathrm{m}}^{(k)}\Pr\left(y_0=\mu\right)\right\}\right\}} \tag{23}$$

for $z_k = \zeta = 1, 2, \ldots, \nu$, where $p_{\mathrm{ij}}^{(k)}$ denotes the element at location i, j in the power matrix P^k and $\pi_{\mathrm{ij}}^{(k)}$ denotes the element at location i, j in the matrix Π_k. The rationale for that approximation is that the resulting algorithm has got the same computational burden as the full information JLQ controller while, partly, accounting for the imperfect information. The gain computations consist of Eqs. (12) - (15) where $\overline{F}$ is replaced by $\overline{\overline{F}}$ from Eq. (23). Notice that the expressions for $\overline{\overline{F}}$ are functions of z_k, where $z_k = 1, 2, \ldots, \nu$. These computations thus involve a set of ν coupled backward matrix equations. These computation are performed off-line and the gains are stored in a look-up table with ν possible values at each step. The resulting control is linear:

$$\mathbf{u}_k = -\Gamma_k\ \mathbf{x}_k \tag{24}$$

As a by-product, the algorithm produces the cost-to-go:

$$J_k^* = \|\mathbf{x}_k\|_{\widetilde{S}_k}^2 + \sum_{i=k}^{N-1} tr\left(W_i S_i\right) \tag{25}$$

since Q_k and R_k are design parameters, they can be chosen via trial and error such as to achieve desired performances. Throughout the paper this algorithm will be addressed as the proposed suboptimal JLQ controller. For the special case where there are no errors in the mode observations, it is straightforward to show that the proposed algorithm boils down to the standard JLQ controller.

3.3 Numerical simulation

The performance of the proposed JLQ control algorithm are illustrated over a simple system via extensive Monte-Carlo simulations. A scalar jump-linear system with three possible values of the mode is considered. The dynamics equation parameters and the cost function weights are given Table 1.

Table 1 System Parameters and Cost Function Weights

y_k	A_k	B_k	Q_k	R_k
1	0.1	1	5	1
2	0.9	2	4	2
3	4	4	2	0.5

The transition probability matrix of the Markov chain y_k is given as follows:

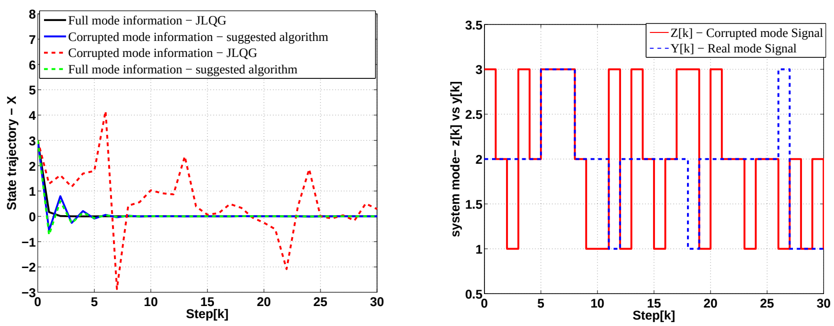
$$P = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{pmatrix} \quad (26)$$

The a priori probabilities for correct and for wrong detections are given as follows:

$$\Pi = \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \quad (27)$$

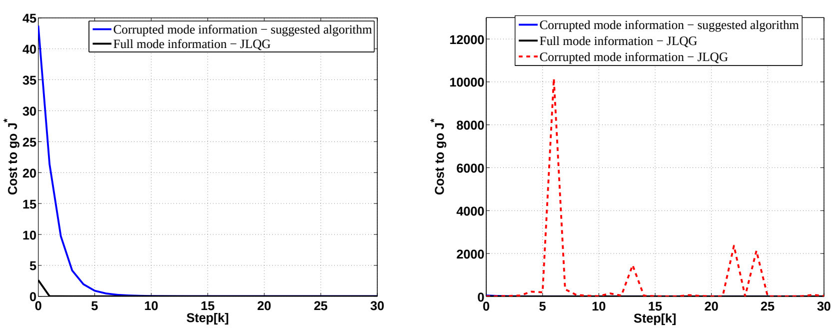
The initial state was chosen to be $x_0 = 3$, the process noise was a zero-mean Gaussian white sequence with a standard deviation of 0.05 and the initial mode distribution was $Pr(y_0) = [1/3, 1/3, 1/3]$. Four simulation cases were examined. In Case 1, there are perfect state and mode measurements and the standard JLQ algorithm is applied. In Case 2, the mode measurement sequence is corrupted, according to the mixing matrix Π in Eq. (27), and the proposed suboptimal JLQ controller is implemented. In Case 3, the novel algorithm is applied in presence of perfect state and mode information. The latter checks the conservativeness of the proposed algorithm. In Case 4, the

standard JLQ algorithm is applied in presence of corrupted mode measurements. This clearly illustrates the motivation for the proposed work. For each case 1000 Monte-Carlo simulation runs were performed along a time space of 30 steps and their averages were computed for each step of the state cost-to-go histories. The results are summarized in Figs. 1 and 2.
Another simulation Case was examined, Case 5, in this case the mode measurement sequence is corrupted and the suboptimal JLQ algorithm based on the increasing mode measurement history is applied (Eq. (19)). Comparing that case to Case 2 show that the novel algorithm achieve fairly close results with much less computational burden. For each case, 1 2 and 5, 10000 Monte-Carlo simulation runs were performed along a short time space of 9 steps, due to the substantial computational burden of Case 5, and their averages were computed for each step of the state. The results are summarized in Fig. 3.



(a) State Trajectories (b) Mode Sequences

Fig. 1 Monte-Carlo means of the state trajectories and an example for the mode sequences.



(a) Cost-to-Go J^* for case 1 and case 2 (b) Cost-to-Go J^* for cases 1,2 and 4

Fig. 2 Monte-Carlo means of the Cost-to-Go J^*.

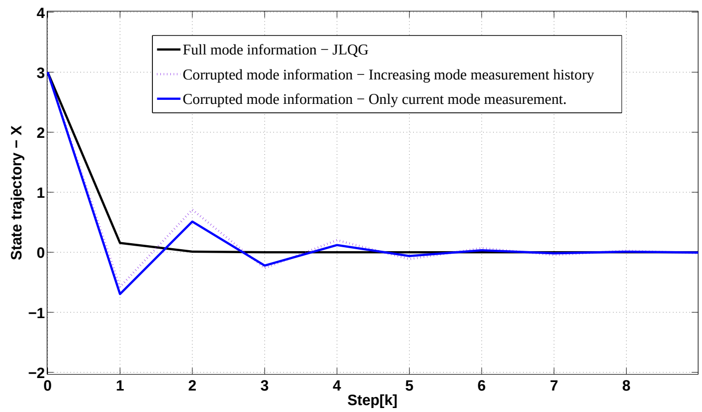


Fig. 3 Monte-Carlo means of the state trajectories.

4 Fault-Tolerant Spacecraft Magnetic Attitude Control

In this section, the efficiency of the proposed fault-tolerant control approach is illustrated on a spacecraft attitude control problem. The case considered is of a very small satellite flying around the earth on a Low-Earth-Orbit, with a mass of a few kilograms. Three Cartesian coordinate reference frames are considered. The first is the inertial ($\mathcal{I}$) frame, its origin is at Earth's center, the z axis coincide with Earth's rotational axis, the x axis point towards a fixed celestial point and y axis is completes a right hand orthogonal frame. The second reference frame is the orbit ($\mathcal{O}$) frame, its origin is at the spacecraft center of mass, the z axis is directed towards nadir, x axis is along the spacecraft orbital velocity vector and y axis also completes a right-handed frame. The third coordinate frame is the body ($\mathcal{B}$) frame, it is also centered at the spacecraft center of mass and its axes coincide with the spacecraft principal axes. The satellite is assumed to be equipped with magnetic actuators only in order to regulate its nadir-pointing attitude. Magnetic actuators, a.k.a magnetorquers, are typical to small spacecraft close enough to the earth such that the magnetic field intensity allows efficient torques. The dynamical model of the satellite is as follows:

$$J\dot{\boldsymbol{\omega}}^{\mathcal{BI}} + \boldsymbol{\omega}^{\mathcal{BI}} \times J\boldsymbol{\omega}^{\mathcal{BI}} = \boldsymbol{m} \times \boldsymbol{b}_{\mathcal{B}} + \boldsymbol{\tau}_g + \boldsymbol{n}_d \tag{28}$$

where $\boldsymbol{\omega}^{\mathcal{B}I}$ denotes the angular velocity of the satellite $\mathcal{B}$ frame with respect to the inertial I frame along the $\mathcal{B}$ frame. $\boldsymbol{\tau}_g$ is the gravity gradient torque expressed in the $\mathcal{B}$ frame, $\boldsymbol{b}_{\mathcal{B}}$ is the body-referenced Earth magnetic field and

$\boldsymbol{m}$ is the body-referenced magnetic control dipole generated by the magnetorquer. $\boldsymbol{n}_d$ is the perturbation torque, it is modeled as a Gaussian zero-mean noise process, it reflects the uncertainty in the satellite model, which is foremost due to residual magnetic torques and atmospheric drag. The quaternion kinematics model is considered as follows,

$$\dot{\boldsymbol{q}} = \frac{1}{2}\Omega \boldsymbol{q} \tag{29}$$

$$\Omega = \begin{pmatrix} 0 & \omega_z^{\mathcal{BO}} & -\omega_y^{\mathcal{BO}} & \omega_x^{\mathcal{BO}} \\ -\omega_z^{\mathcal{BO}} & 0 & \omega_x^{\mathcal{BO}} & \omega_y^{\mathcal{BO}} \\ \omega_y^{\mathcal{BO}} & -\omega_x^{\mathcal{BO}} & 0 & \omega_z^{\mathcal{BO}} \\ -\omega_x^{\mathcal{BO}} & -\omega_y^{\mathcal{BO}} & -\omega_z^{\mathcal{BO}} & 0 \end{pmatrix} \tag{30}$$

where $\boldsymbol{\omega}^{\mathcal{BO}}$ denotes the body-referenced angular velocity of $\mathcal{B}$ with respect to $\mathcal{O}$. Using rigid body assumptions and small angle approximations, the dynamics and the kinematics modeling equations lend themselves to a set of linear differential equations [12],

$$\dot{\mathbf{x}} = A\mathbf{x} + B(t)\,\mathbf{m} + G\,\mathbf{n}_d \tag{31}$$

And it is expressed as follows:,

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\omega}_x^{\mathcal{BO}} \\ \dot{\omega}_y^{\mathcal{BO}} \\ \dot{\omega}_z^{\mathcal{BO}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -4\omega_{or}^2\sigma_1 & 0 & 0 & 0 & 0 & \omega_{or}\,(1-\sigma_1) \\ 0 & 3\omega_{or}^2\sigma_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega_{or}^2\sigma_3 & -\omega_{or}\,(1+\sigma_3) & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \theta \\ \psi \\ \omega_x^{\mathcal{BO}} \\ \omega_y^{\mathcal{BO}} \\ \omega_z^{\mathcal{BO}} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_z}{J_{11}} & -\frac{b_y}{J_{11}} \\ -\frac{b_z}{J_{22}} & 0 & \frac{b_x}{J_{22}} \\ \frac{b_y}{J_{33}} & -\frac{b_x}{J_{33}} & 0 \end{pmatrix} \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{J_{11}} & 0 & 0 \\ 0 & \frac{1}{J_{22}} & 0 \\ 0 & 0 & \frac{1}{J_{33}} \end{pmatrix} \boldsymbol{n}_d \tag{32}$$

$$\sigma_1 = \frac{J_{22} - J_{33}}{J_{11}} \tag{33}$$

$$\sigma_2 = \frac{J_{33} - J_{11}}{J_{22}} \tag{34}$$

$$\sigma_3 = \frac{J_{11} - J_{22}}{J_{33}} \tag{35}$$

The angles ϕ, θ and ψ are the roll, pitch, and yaw Euler angles, respectively. The magnetic control torques are proportional to the magnetic field components and to the on-board magnetic dipole components (with bounds of $1\,Am^2$). The magnetic field along the trajectory, the attitude, and the attitude rates are assumed known. Three types of modes are considered. The first mode is the "healthy" mode where no failures of the magnetometers nor disturbances in the magnetorquers occur. In the second mode, the magnetic rods are assumed to loose efficiency due to a loss of voltage. The logic of this mode consists in maintaining an attitude stabilization capability to some extent, while saving power. The third mode happens when the magnetometers fail to sense the magnetic field by loosing 90% of their output signal strength. These three modes, i.e., the healthy and the two failed modes, are assumed to occur at random according to the next transition probabilities,

$$P = \begin{pmatrix} 0.985 & 0.01 & 0.005 \\ 0.02 & 0.975 & 0.005 \\ 0.02 & 0.02 & 0.96 \end{pmatrix} \tag{36}$$

and their occurring is detected with same correct and wrong a priori probabilities as in Eq. (27). The weighing matrices, $Q(y)$ and $R(y)$, were tuned such as to fit with the model assumptions and to yield best possible performances. An example for associated parameters for the modes are displayed in Table 2. The dynamical state was initialized with, $\{20[deg]; -15[deg]; 15[deg]; 1\times 10^{-3}[\frac{rad}{sec}]; 5\times 10^{-4}[\frac{rad}{sec}]; -1.2\times 10^{-3}[\frac{rad}{sec}]\}$, and it was assumed that the initial mode was chosen at random in each run using the steady-state distribution $\{0.571,\ 0.317,\ 0.111\}$. Extensive Monte-Carlo simulations were run in order to illustrate the proposed regulator performances. Four cases were tested. Case 1 consists in applying the JLQ regulator of Section 1 when the mode is perfectly known. Case 2 consists in applying the proposed fault-tolerant JLQ regulator when the mode is measured with errors. In Case 3, the fault-tolerant JLQ algorithm is applied while the mode is perfectly measured, and in Case 4 the original JLQ is applied while the mode is imperfectly measured. The results are summarized in Figs. 4 and 5. The plots depict the time histories of the pointing errors for all four cases. Figure 4 shows that the applied JLQ magnetic attitude controllers succeed in achieving a nadir-pointing accuracy of the order of 1 degree in steady state for Cases 1, 2, and 3, and that cases 2 and 3 show relatively close performance. Figure 5 illustrates the fact that the original JLQ attitude controller can not handle imperfect mode measurements.

Table 2 Modes and Parameters

y_k	$B_k(y_k)$	$Q_k(y_k)$	$R_k(y_k)$
Ideal	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_z}{J_{11}} & -\frac{b_y}{J_{11}} \\ -\frac{b_z}{J_{22}} & 0 & \frac{b_x}{J_{22}} \\ \frac{b_y}{J_{33}} & -\frac{b_x}{J_{33}} & 0 \end{pmatrix}$	$\begin{pmatrix} 38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{pmatrix}$	$\begin{pmatrix} 2.5 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 10 \end{pmatrix}$
Power saving	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{b_z}{J_{11}} & -\frac{b_y}{J_{11}} \\ -\frac{b_z}{J_{22}} & 0 & \frac{b_x}{J_{22}} \\ \frac{b_y}{J_{33}} & -\frac{b_x}{J_{33}} & 0 \end{pmatrix}$	$\begin{pmatrix} 38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{pmatrix}$	$\begin{pmatrix} 300 & 0 & 0 \\ 0 & 800 & 0 \\ 0 & 0 & 400 \end{pmatrix}$
90% *decrease*	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{0.1b_z}{J_{11}} & -\frac{0.1b_y}{J_{11}} \\ -\frac{0.1b_z}{J_{22}} & 0 & \frac{0.1b_x}{J_{22}} \\ \frac{0.1b_y}{J_{33}} & -\frac{0.1b_x}{J_{33}} & 0 \end{pmatrix}$	$\begin{pmatrix} 38 & 0 & 0 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1 \end{pmatrix}$	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 10 \end{pmatrix}$

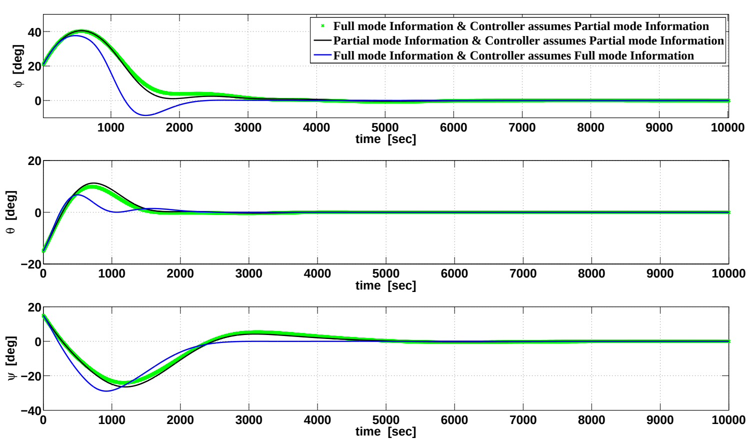


Fig. 4 Pointing errors for Cases 1 to 3. Monte-Carlo averages (50).

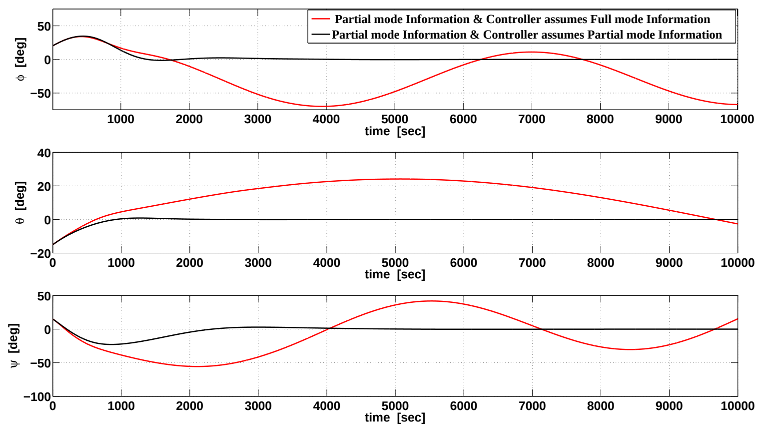


Fig. 5 Pointing errors in Cases 2 and 4. Monte-Carlo averages (50).

5 Conclusion

In this work, a novel suboptimal JLQ controller is suggested for discrete-time dynamical systems under the assumptions of full state information and a priori probabilities for correct and for wrong mode detections. Comparative results of an extensive Monte-Carlo simulation for a simple system illustrates the efficiency and conservativeness of the proposed algorithm that mitigates the destabilizing effect of corrupted mode observations while having the same computational burden as the full information JLQ controller. The fault-tolerant spacecraft attitude control results illustrate the validity of the approach, showing a steady state pointing accuracy of a few degrees, which is typical for magnetic based small satellites control performances. This general approach shows flexibility from the modeling standpoint and proves to be promising for the development of successful fault-tolerant attitude controllers.

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Appendix

A. Recursive formulation of the control problem via Dynamic Programming:

Given the cost function,

$$J = E\left\{\sum_{k=0}^{N-1} \|x_{k+1}\|_{Q_k}^2 + \|u_k\|_{R_k}^2\right\} \tag{37}$$

Consider the problem of minimizing J with respect to $\{\mathbf{u}_k\}$,

$$J_k^*\left(\mathcal{X}^k, \mathcal{Z}^k\right) \triangleq \min_{U_k^{N-1}} E\left\{\sum_{i=k}^{N-1} \|x_{i+1}\|_{Q_i}^2 + \|u_i\|_{R_i}^2 \,|\mathcal{X}^k, \mathcal{Z}^k\right\} \tag{38}$$

where $\mathcal{X}^k \triangleq \{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k\}$ and $\mathcal{Z}^k \triangleq \{z_0, z_1, \ldots, z_k\}$ denote the histories of $\{\mathbf{x}_k\}$ and $\{\mathbf{z}_k\}$, respectively.

Recursive formulation of the control problem:

Start with step $N-1$:

$$\min_{u_{N-1}} J_{N-1}^* = \min_{u_{N-1}} E\left\{\|x_N\|_{Q_{N-1}}^2 + \|u_{N-1}\|_{R_{N-1}}^2 |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}$$

$$x_N = A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1}$$

$$\begin{aligned}
\min_{u_{N-1}} J_{N-1}^* = \min_{u_{N-1}} & E\{\|A_{N-1}x_{N-1} + B_{N-1}u_{N-1} + w_{N-1}\|_{Q_{N-1}}^2 + \\
& + \|u_{N-1}\|_{R_{N-1}}^2 |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\} \\
= \min_{u_{N-1}} & \{\|u_{N-1}\|_{E\left\{B_{N-1}^T Q_{N-1} B_{N-1} + R_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}}^2 \\
& + \|x_{N-1}\|_{E\left\{A_{N-1}^T Q_{N-1} A_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}}^2 \\
& + 2x_{N-1}^T E\left\{A_{N-1}^T Q_{N-1} B_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\} u_{N-1} \\
& + 2x_{N-1}^T \underbrace{E\left\{A_{N-1}^T Q_{N-1} w_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}}_{=0} \\
& + 2u_{N-1}^T \underbrace{E\left\{B_{N-1}^T Q_{N-1} w_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}}_{=0} \\
& + \underbrace{E\left\{w_{N-1}^T Q_{N-1} w_{N-1} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}}_{\mathrm{Trace}\{W_{N-1} Q_{N-1}\}}\}
\end{aligned}$$

$$\min_{u_{N-1}} J^*_{N-1} = \min_{u_{N-1}} \{\mathrm{tr}\{W_{N-1}Q_{N-1}\} +$$
$$\left\|x_{N-1}\right\|^2_{\overline{A^T_{N-1}Q_{N-1}A_{N-1}} - \overline{A^T_{N-1}Q_{N-1}B_{N-1}}\left[\overline{B^T_{N-1}Q_{N-1}B_{N-1}+R_{N-1}}\right]^{-1}\overline{B^T_{N-1}Q_{N-1}A_{N-1}}}$$
$$+ \left\|u_{N-1} + \left[\overline{B^T_{N-1}Q_{N-1}B_{N-1} + R_{N-1}}\right]^{-1} \times \overline{B^T_{N-1}Q_{N-1}A_{N-1}}x_{N-1}\right\|^2_{\overline{B^T_{N-1}Q_{N-1}B_{N-1}+R_{N-1}}}\}$$

And for summary the solution for step $N-1$ is,

$$u^*_{N-1} = -\left[\overline{B^T_{N-1}Q_{N-1}B_{N-1} + R_{N-1}}\right]^{-1}\overline{B^T_{N-1}Q_{N-1}A_{N-1}}x_{N-1}$$
$$u^*_{N-1} = -\Gamma_{N-1}x_{N-1}$$
$$J^*_{N-1} = \|x_{N-1}\|^2_{\overline{A^T_{N-1}Q_{N-1}A_{N-1}} - \overline{A^T_{N-1}Q_{N-1}B_{N-1}}\Gamma_{N-1}} + \mathrm{tr}\{W_{N-1}Q_{N-1}\}$$
$$J^*_{N-1} = \|x_{N-1}\|^2_{\tilde{S}_{N-1}} + \mathrm{tr}\{W_{N-1}Q_{N-1}\}$$

Step $N-2$:

$$J^*_{N-2}\left(\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right) \equiv \min_{U^{N-1}_{N-2}} E\left\{\sum_{i=N-2}^{N-1} \|x_{i+1}\|^2_{Q_i} + \|u_i\|^2_{R_i} |\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right\}$$
$$= \min_{U^{N-1}_{N-2}} E\left\{\|x_N\|^2_{Q_{N-1}} + \|u_{N-1}\|^2_{R_{N-1}} + \|x_{N-1}\|^2_{Q_{N-2}} + \|u_{N-2}\|^2_{R_{N-2}} |\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right\}$$
$$= \min_{u_{N-2}} E\{\|x_{N-1}\|^2_{Q_{N-2}} + \|u_{N-2}\|^2_{R_{N-2}} + \min_{u_{N-1}} E\left\{\|x_N\|^2_{Q_{N-1}} + \|u_{N-1}\|^2_{R_{N-1}} |\mathcal{X}^{N-1}, \mathcal{Z}^{N-1}\right\}|\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\Big\}$$
$$= \min_{u_{N-2}} E\left\{\|x_{N-1}\|^2_{Q_{N-2}} + \|u_{N-2}\|^2_{R_{N-2}} + J^*_{N-1}|\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right\}$$
$$= \min_{u_{N-2}} E\left\{\|x_{N-1}\|^2_{Q_{N-2}} + \|u_{N-2}\|^2_{R_{N-2}} + \|x_{N-1}\|^2_{\tilde{S}_{N-1}} + \mathrm{tr}\{W_{N-1}Q_{N-1}\} |\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right\}$$
$$= \min_{u_{N-2}} E\left\{\|x_{N-1}\|^2_{Q_{N-2}+\tilde{S}_{N-1}} + \|u_{N-2}\|^2_{R_{N-2}} |\mathcal{X}^{N-2}, \mathcal{Z}^{N-2}\right\} + \mathrm{tr}\{W_{N-1}Q_{N-1}\}$$

The solution for step $N-2$ is,

$$u^*_{N-2} = -\left[\overline{B^T_{N-2}\left(Q_{N-2}+\tilde{S}_{N-1}\right)B_{N-2}+R_{N-2}}\right]^{-1} \times \overline{B^T_{N-2}\left(Q_{N-2}+\tilde{S}_{N-1}\right)A_{N-2}}x_{N-2}$$

$$u^*_{N-2} = -\Gamma_{N-2}x_{N-2}$$

$$J^*_{N-2} = \|x_{N-2}\|^2_{\overline{A^T_{N-2}(Q_{N-2}+\tilde{S}_{N-1})A_{N-2}}-\overline{A^T_{N-2}(Q_{N-2}+\tilde{S}_{N-1})B_{N-2}}\Gamma_{N-2}} + \mathrm{tr}\left\{W_{N-1}Q_{N-1}\right\} + \mathrm{tr}\left\{W_{N-2}\left(Q_{N-2}+\tilde{S}_{N-1}\right)\right\}$$

$$= \|x_{N-1}\|^2_{\overline{A^T_{N-2}S_{N-2}A_{N-2}}-\overline{A^T_{N-2}S_{N-2}B_{N-2}}\Gamma_{N-2}} + \sum_{i=N-2}^{N-1}\mathrm{tr}\left\{W_iS_i\right\}$$

It is straightforward to show that the solution for step k is,

$$u^*_k = -\left[\overline{B^T_k\left(Q_k+\tilde{S}_{k+1}\right)B_k+R_k}\right]^{-1}\overline{B^T_k\left(Q_k+\tilde{S}_{k+1}\right)A_k}x_k = -\Gamma_kx_k \tag{39}$$

$$J^*_k = \|x_k\|^2_{\overline{A^T_kS_kA_k}-\overline{A^T_kS_kB_k}\Gamma_k} + \sum_{i=k}^{N-1}\mathrm{tr}\left\{W_iS_i\right\} = \|x_k\|^2_{\tilde{S}_k} + \sum_{i=k}^{N-1}\mathrm{tr}\left\{W_iS_i\right\} \tag{40}$$

Where, for $k=N$ initialize the computation with

$$\tilde{S}_N = 0 \tag{41}$$

For $k = N-1, N-2, ..., 0$, compute

$$S_k\left(\mathcal{X}^{k+1},\mathcal{Z}^{k+1},y_k\right) = \tilde{S}_{k+1}+Q_k \tag{42}$$

$$\Gamma_k\left(\mathcal{X}^k,\mathcal{Z}^k\right) = \left(\overline{B^T_kS_kB_k+R_k}\right)^{-1}\overline{B^T_kS_kA_k} \tag{43}$$

$$\tilde{S}_k\left(\mathcal{X}^k,\mathcal{Z}^k\right) = \overline{A^T_kS_kA_k} - \overline{A^T_kS_kB_k}\Gamma_k \tag{44}$$

The variables $\overline{F}$ in Eqs. (43) and (44) are defined as follows:

$$\overline{F}(\mathcal{X}^k,\mathcal{Z}^k) \triangleq E\{F \mid \mathcal{X}^k,\mathcal{Z}^k\} \tag{45}$$

B. Conditional Expectation Approximations

In the case examined in this work there is a need to calculate a conditional expectation of the next form,

$$\overline{F}(\mathcal{X}^k, \mathcal{Z}^k) \stackrel{\triangle}{=} E\{F \mid \mathcal{X}^k, \mathcal{Z}^k\} \tag{46}$$

Considering the conditioning sequence in Eq. (46) the computation of these conditional expectations would require a growing memory size. Two approximate solutions are examined, the first approximation is the case where only the mode observation histories $\mathcal{Z}^k$ are retained, i.e., by computing the following conditional expectation

$$E\left\{F|\mathcal{X}^k, \mathcal{Z}^k\right\} \approx \tilde{\overline{F}}(\mathcal{Z}^k) = E\{F \mid \mathcal{Z}^k\} = E\left\{F\left(y_k, \mathcal{Z}^k, z_{k+1}\right)|\mathcal{Z}^k\right\} \tag{47}$$

$$\tilde{\overline{F}}(\mathcal{Z}^k) = \sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y} F\left(y_k, \mathcal{Z}^k, z_{k+1}\right)\Pr\left(y_k, z_{k+1}|\mathcal{Z}^k\right) \tag{48}$$

In order to find the expression for the conditional probability $\Pr\left(y_k, z_{k+1}|\mathcal{Z}^k\right)$ we will first examine the analytic solution from step k=0,

$$\Pr\left(y_0, z_1|\mathcal{Z}^0\right) = \frac{\Pr\left(y_0, z_1, \mathcal{Z}^0\right)}{\Pr\left(\mathcal{Z}^0\right)} = \frac{\Pr\left(y_0, z_1, z_0\right)}{\Pr\left(z_0\right)}$$
$$\Pr\left(y_0, z_1|\mathcal{Z}^0\right) = \frac{\Pr\left(z_1|y_0, z_0\right)\Pr\left(z_0|y_0\right)\Pr\left(y_0\right)}{\Pr\left(z_0\right)}$$
$$\Pr\left(y_0, z_1|\mathcal{Z}^0\right) = \frac{\sum_{y_1}^{\nu}\left\{\Pr\left(z_1|y_1\right)\Pr\left(y_1|y_0\right)\right\}\Pr\left(z_0|y_0\right)\Pr\left(y_0\right)}{\Pr\left(z_0\right)}$$

For step $k = 1$,

$$Pr\left(y_1, z_2|\mathcal{Z}^1\right) = \frac{\Pr\left(y_1, z_2, \mathcal{Z}^1\right)}{\Pr\left(\mathcal{Z}^1\right)} = \frac{\Pr\left(y_1, z_0, z_1, z_2\right)}{\Pr\left(z_0, z_1\right)}$$
$$\Pr\left(y_1, z_2|\mathcal{Z}^1\right) = \frac{\Pr\left(z_2|y_1, z_0, z_1\right)\sum_{y_0}^{\nu}\Pr\left(y_0, y_1, z_0, z_1\right)\Pr\left(y_0\right)}{\Pr\left(z_0, z_1\right)}$$
$$= \frac{\Pr\left(z_2|y_1, z_0, z_1\right)\sum_{y_0}^{\nu}\left\{\Pr\left(y_1|y_0, z_0, z_1\right)\Pr\left(y_0\right)\Pr\left(y_0, z_1, z_0\right)\right\}}{\Pr\left(z_0, z_1\right)}$$
$$= \frac{\Pr\left(z_2|y_1, z_0, z_1\right)\sum_{y_0}^{\nu}\left\{\Pr\left(y_1|y_0\right)\Pr\left(y_0\right)\Pr\left(y_0, z_1|\mathcal{Z}^0\right)\Pr\left(z_0\right)\right\}}{\sum_{y_0}^{\nu}\Pr\left(y_0, z_0, z_1\right)\Pr\left(y_0\right)}$$
$$= \frac{\sum_{y_2}^{\nu}\left\{\Pr\left(z_2|y_2\right)\Pr\left(y_2|y_1\right)\right\}\sum_{y_0}^{\nu}\left\{\Pr\left(y_1|y_0\right)\Pr\left(y_0\right)\Pr\left(y_0, z_1|\mathcal{Z}^0\right)\Pr\left(z_0\right)\right\}}{\sum_{y_0}^{\nu}\left\{\Pr\left(y_0, z_1|\mathcal{Z}^0\right)\Pr\left(z_0\right)\Pr\left(y_0\right)\right\}}$$
$$= \frac{\sum_{y_2}^{\nu}\left\{\Pr\left(z_2|y_2\right)\Pr\left(y_2|y_1\right)\right\}\sum_{y_0}^{\nu}\left\{\Pr\left(y_1|y_0\right)\Pr\left(y_0\right)\Pr\left(y_0, z_1|\mathcal{Z}^0\right)\right\}}{\sum_{y_0}^{\nu}\left\{\Pr\left(y_0, z_1|\mathcal{Z}^0\right)\Pr\left(y_0\right)\right\}}$$

And for step $k = 2$,

$$\Pr\left(y_2, z_3|\mathcal{Z}^2\right) = \frac{\Pr\left(y_2, z_3, \mathcal{Z}^2\right)}{\Pr\left(\mathcal{Z}^2\right)} = \frac{\Pr\left(y_2, z_3, z_2, z_1, z_0\right)}{\Pr\left(z_2, \mathcal{Z}^1\right)}$$

$$\Pr\left(y_2, z_3|\mathcal{Z}^2\right) = \frac{\Pr\left(z_3|y_2, \mathcal{Z}^2\right)\sum_{y_1}^{\nu}\left\{\Pr\left(y_2, y_1, z_2, \mathcal{Z}^1\right)\Pr\left(y_1\right)\right\}}{\Pr\left(z_2, \mathcal{Z}^1\right)}$$

$$= \frac{\sum_{y_3}^{\nu}\left\{\Pr\left(z_3|y_3\right)\Pr\left(y_3|y_2\right)\right\}\sum_{y_1}^{\nu}\left\{\Pr\left(y_2|y_1\right)\Pr\left(y_1\right)\Pr\left(y_1, z_2|\mathcal{Z}^1\right)\Pr\left(\mathcal{Z}^1\right)\right\}}{\sum_{y_1}^{\nu}\left\{\Pr\left(y_1, z_2|\mathcal{Z}^1\right)\Pr\left(\mathcal{Z}^1\right)\Pr\left(y_1\right)\right\}}$$

$$= \frac{\sum_{y_3}^{\nu}\left\{\Pr\left(z_3|y_3\right)\Pr\left(y_3|y_2\right)\right\}\sum_{y_1}^{\nu}\left\{\Pr\left(y_2|y_1\right)\Pr\left(y_1\right)\Pr\left(y_1, z_2|\mathcal{Z}^1\right)\right\}}{\sum_{y_1}^{\nu}\left\{\Pr\left(y_1, z_2|\mathcal{Z}^1\right)\Pr\left(y_1\right)\right\}}$$

It is straightforward to show that the solution for step $k > 0$ is,

$$\Pr\left(y_k, z_{k+1}|\mathcal{Z}^k\right) = \sum_{y_{k+1}}^{\nu}\left\{\Pr\left(z_{k+1}|y_{k+1}\right)\Pr\left(y_{k+1}|y_k\right)\right\} \times \frac{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_k|y_{k-1}\right)\Pr\left(y_{k-1}, z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}}{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_{k-1}, z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}}$$

The computations of $\tilde{\overline{F}}(\mathcal{Z}^k)$ are performed as follows:

$$\tilde{\overline{F}}(\mathcal{Z}^k) = \sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y} F\left(y_k, \mathcal{Z}^k, z_{k+1}\right)\sum_{y_{k+1}}^{\nu}\left\{\Pr\left(z_{k+1}|y_{k+1}\right)\Pr\left(y_{k+1}|y_k\right)\right\} \times \frac{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_k|y_{k-1}\right)\Pr\left(y_{k-1}, z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}}{\sum_{y_{k-1}}^{\nu}\left\{\Pr\left(y_{k-1}, z_k|\mathcal{Z}^{k-1}\right)\Pr\left(y_{k-1}\right)\right\}}$$

The second suggested approximate solution is achieved by retaining only the current mode observation z_k, i.e., by computing the following conditional expectation:

$$E\left\{F|\mathcal{X}^k, \mathcal{Z}^k\right\} \approx \overline{\overline{F}}(z_k) = E\{F \mid z_k\} = E\left\{F\left(y_k, z_{k+1}\right)|z_k\right\} \tag{49}$$

$$\overline{\overline{F}}(z_k) = \sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y} F\left(y_k, z_{k+1}\right)\Pr\left(y_k, z_{k+1}|z_k\right) \tag{50}$$

The computations of $\Pr\left(y_k, z_{k+1}|z_k\right)$ are performed as follows:

$$\Pr\left(y_k, z_{k+1}|z_k\right) = \frac{\Pr\left(y_k, z_{k+1}, z_k\right)}{\Pr\left(z_k\right)} = \frac{\Pr\left(z_{k+1}|y_k, z_k\right)\Pr\left(z_k|y_k\right)\Pr\left(y_k\right)}{\Pr\left(z_k\right)}$$

$$\Pr\left(y_k, z_{k+1}|z_k\right) = \frac{\sum_{y_{k+1}}^{\nu}\left\{\Pr\left(z_{k+1}|y_{k+1}\right)\Pr\left(y_{k+1}|y_k\right)\right\}\Pr\left(z_k|y_k\right)\Pr\left(y_k\right)}{\Pr\left(z_k\right)}$$

The computations of $\overline{\overline{F}}(z_k)$ are performed as follows:

$$\overline{\overline{F}}(z_k) = \sum_{z_{k+1}\in S_z}\sum_{y_k\in S_y} F(y_k, z_{k+1}) \times \frac{\Pr(z_k|y_k)\Pr(y_k)\sum_{y_{k+1}\in S_y}\{\Pr(z_{k+1}|y_{k+1})\Pr(y_{k+1}|y_k)\}}{\sum_{y_k\in S_y}\{\Pr(z_k|y_k)\Pr(y_k)\}} \tag{51}$$

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