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Nonlinear Output-Feedback H_∞ Control for Spacecraft Attitude Control

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Abstract
In this paper, a novel computational scheme is proposed in order to solve the output-feedback H_∞ control problem for a class of nonlinear systems with polynomial vector field. By converting the resulting Hamilton-Jacobi inequalities from rational forms to their equivalent polynomial forms, we overcome the non-convex nature and numerical difficulty. Using quadratic Lyapunov functions, both the state-feedback and output-feedback problems are reformulated as semi-definite optimization conditions, while locally tractable solutions can be obtained through sum of squares (SOS) programming. A numerical example shows that the proposed computational scheme results in a better disturbance attenuation closed-loop system, as compared to standard methods, by using classical quadratic Lyapunov functions. The novel methodology is applied in order to develop a robust spacecraft attitude regulator.

In the past decade, there has been substantial interest in H_∞ control of nonlinear systems [23, 8]. Interpreting nonlinear H_∞ control in terms of dissipativity and differential game [2, 24] where the solution has been related to an appropriate Hamilton-Jacobi inequality. For hyperbolic nonlinear systems whose linearized

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Nadav Berman passed away on April 21, 2012. Professor Berman was a very active member of the Israeli community of control scientists he was an admired figure and an exceptional supervisor who we dearly miss.
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plant is stabilizable, the solution of the Hamilton-Jacobi inequality was characterized by an invariant manifold of Hamiltonian vector fields using differential geometric theory [23, 24].

In linear systems, it is well known that the Hamilton-Jacobi partial differential inequality reduces to the Riccati inequality, which can be solved easily by efficient numerical algorithms. However, in the nonlinear case, there is no systematic numerical algorithm currently available for the solution of this partial differential inequality. Therefore, the key difficulty of nonlinear H_∞ control theory is the solvability of the Hamilton-Jacobi inequality. To this end, various approaches have been proposed to solve the Hamilton-Jacobi inequality numerically. One of the suggested methods is a Taylor series expansion of the storage function [10, 29], in an iterative fashion, provided that the linearized model of the nonlinear system has a solution. However, a numerically efficient solution remains an unsolved issue [1].

Isidori [8] showed, that the solution to the output-feedback control problem is determined by a pair of coupled Hamilton-Jacobi inequalities. Parallel to linear H_∞ control theory, a separation principle was also established under a detectability hypothesis [9]. Obviously, there are major advantages of the output-feedback problem for continuous-time nonlinear systems over linear systems [1], despite the fact that the output-feedback problem for nonlinear systems has not been studied as widely as for linear systems. Although there are studies of the static output-feedback for nonlinear systems, the dynamic output-feedback for nonlinear systems was studied much less; one of the reasons is the non-complex structure rather than the dynamic output-feedback case. In addition, it preserves the controllers structure, based on the physical intuition from the actual system. Yet, the dynamic output-feedback results in high order controllers [8] which are more accurate. The dynamic output-feedback problem has been investigated while parameterized as a nonlinear fractional transformation on locally contractive and stable nonlinear operators [12]. A solution based on allowing nonhyperbolic equilibria for the Hamiltonian systems associated with the two Hamilton-Jacobi-Isaacs equations: the state-feedback and, respectively, output-injection design problems are presented in [8, 25]. However, the solutions from these approaches do not have a closed form and therefore may not converge to an analytic solution, due to their non-convex nature.

A recent computational relaxation based on the sum of squares (SOS) decomposition for multivariable polynomials and semidefinite programming [16, 4] provides potentially effective ways for the analysis and synthesis of nonlinear systems. In nonlinear system design, the verification of the non-negativity of the Lyapunov conditions is a complex task. However, the new computationally tractable analysis methodology provides a new way of searching for SOS decomposition to relax the original problem. This crucial property of the SOS based methodology finds applications successfully in several nonlinear control problems. For example, the stability analysis and synthesis problem have been studied in [19, 3, 18, 27] for nonlinear systems. In [31] local stability analysis was considered, and the region of attraction inner-bound enlargement problem was presented for polynomial systems with uncertain dynamics. A semidefinite programming approach based on state dependent inequalities is proposed in [17] to obtain global stability and performance objective by using quadratic Lyapunov functions.

As a result, a convex parametrization of the nonlinear H_∞ control problem was derived in [13] based on a pair of positive definite matrix functions. Prempain [21] formulated the $\mathscr{L}_2$-gain analysis problem for polynomial nonlinear systems as a convex state-dependent LMI, which can be recasted as a SOS optimization problem. This approach was shown promising to overcome the numerical difficulty in solving the Hamilton-Jacobi inequality and provides an analytic solution at the same time. Wei *et al.* [32] proposed an iterative method based on SOS programming [18], [7] to solve a special case of the state-feedback H_∞ control problem. As a powerful and promising technique, SOS programming has also been applied to solve nonlinear analysis [15], [28] and stabilization [17], [20] problems. The main advantages of SOS decomposition are the resulting computational tractability and the algorithmic characteristics of the solution procedure [16]. This could help to provide coherent methodology of synthesizing Lyapunov functions for nonlinear systems. In addition, the importance of SOS technique also lies in its ability to provide tractable relaxations for many difficult optimization problems, such as the nonlinear output-feedback H_∞ controller.

Motivated by all of these developments, we propose a computational scheme for solving the nonlinear dynamic output-feedback design problems for a class of affine nonlinear systems. Moreover, the resulting output feedback controller will be constructed to achieve closed-loop stability as well as $\mathscr{L}_2$-gain performance. Specifically, we use polynomial type Lyapunov functions to convert the original Hamilton-Jacobi inequalities into linear matrix inequalities for polynomial nonlinear systems. As a result, the numerical difficulty in solving the nonlinear H_∞ output-feedback problem is overcome, and the output-feedback controllers and Lyapunov functions are constructed in an efficient computational manner.

Spacecraft attitude control is a critical function in any space mission. The development of nonlinear spacecraft attitude control algorithms has been following many paths over the last four decades, from Lyapunov-based regulator [14], nonlinear adaptive control [22], dynamic inversion [5], optimization [26], model predictive control [11], to sliding mode control, State-Dependent-Riccati-Equation (SDRE) control [6], and H_∞ control [30]. Applying the proposed novel methodology, a robust attitude controller will be developed in the final manuscript under the assumptions of rigid body dynamics, three-axis control authority, and full state information. Using the quaternion of rotation and the angular velocity vector as state variables yields a polynomial structure of the dynamical model, enabling the novel H_∞ control design. Particular attention will be given to the quaternion properties, i.e., non-uniqueness with regard to attitude and norm unity.

1 Sum Of Squares

A basic problem that appears in many areas of mathematics is that of checking global non-negativity of a function of several variables. In particular, the problem is to establish equivalent conditions or a procedure for checking the validity of:

$$F(x_1,\dots,x_n) \geq 0, \quad \forall x_1,\dots,x_n \in \mathbb{R} \tag{1}$$

A polynomial $F(x) \in \mathbb{R}[x]$ is said to be nonnegative or positive semidefinite (PSD) if $F(x) \geq 0 \; \forall \; x \in \mathbb{R}^n$. Clearly, a necessary condition for a polynomial to be PSD is that its total degree be even. We say that $F(x)$ is sum of squares (SOS), if there exist polynomials $f_1(x),\dots f_m(x)$ such that:

$$F(x) = \sum_{i=1}^{m} f_i^2(x) \tag{2}$$

It is clear that $F(x)$ being SOS implies that $F(x)$ is PSD. We define a function $q : \mathbb{R}^n \to \mathbb{R}$ as a *monomial* if:

$$q(x) = c_a x^a, \quad x \in \mathbb{R}^n, c_a \in \mathbb{R}, a \in \mathbb{N}^n \tag{3}$$

such that $q(x) = c_a\left(x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}\right)$. Defining a function $p = \sum_{i=1}^{r} q_i(x)$ to be polynomial if it is a sum of monomials $q_1, q_2, \dots, q_r : \mathbb{R}^n \to \mathbb{R}$ with finite degree. The largest degree of the monomials $q_1, q_2, \dots, q_r$ is defined to be the degree of p. A set of polynomials $p : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\mathscr{P}$, where the polynomial with the largest degree defines the degree of the family $\mathscr{P}$. We define $x^{\{d\}} \in \mathbb{R}^{\sigma(n,d)}$ with $x \in \mathbb{R}^n$ as a *vector of monomials* for the polynomials in $\mathscr{P}$ of degree d, as a basis of $\mathscr{P}$, where $\sigma(n,d)$ is defined as, $\sigma(n,d) = \frac{(n+d-1)!}{(n-1)!d!}$ in n scalar variables. The basic idea of the method is the following: express the given polynomial as a quadratic form in some new variables $x^{\{d\}}$. These new variables are the original x ones, plus all the monomials of degree less than or equal to $\frac{d}{2}$, given by the different products of the x variables, where d is the degree of the polynomial. Therefore, $F(x)$ can be represented as:

$$F(x) = x^{\{d\}^T} Q x^{\{d\}} \tag{4}$$

where Q is a constant matrix called the Gramian matrix, not necessarily unique. The following representation is also called the square matrix representation (SMR). If in the representation above Q is positive semidefinite, then $F(x)$ is also positive semidefinite. Notice that in the case of quadratic forms, for instance, the two conditions (nonegativity and sum of squares) are equivalent. The problem of checking if a given polynomial may be written as a sum of squares can be solved via convex optimization, in particular semidefinite programming. SOSTOOLS a free, third party MATLAB toolbox provides a way of finding sum of squares, over an affine family of polynomials. For instance, it can be used in the computation of Lyapunov functions for proving stability of nonlinear systems.

2 The Nonlinear H_∞ Problem

Considering the following system where the plant is represented by an affine causal state space system defined on a smooth n-dimensional manifold $\mathscr{X} \subseteq \mathbb{R}^n$ in local coordinates $x = (x_1, \ldots, x_n)$:

$$\Sigma: \begin{array}{l} \dot{x} = f(x) + g_1(x)w + g_2(x)u \\ y = x \\ z = h_1(x) + k_{12}(x)u, \quad z \in \mathbb{R}^s \end{array} \tag{5}$$

with two sets of inputs u and w and two sets of outputs y and z. Where $x \in \mathscr{X}$ is the state vector, $u \in \mathscr{U} \subseteq \mathbb{R}^p$ is the p-dimensional control input, which belongs to the set of admissible controls $\mathscr{U}$, $w \in \mathscr{W}$ is the disturbance signal, which belongs to the set $\mathscr{W} \subset \mathscr{L}_2([t_0,\infty),\mathbb{R}^r)$ of admissible disturbances. The output $y \in \mathbb{R}^n$ is the states vector of the system which is measured directly, and $z \in \mathbb{R}^s$ is the output to be controlled. The functions $f : \mathscr{X} \to C^\infty(\mathscr{X})$, $g_1 : \mathscr{X} \to \mathscr{M}^{n\times r}(\mathscr{X})$, $g_2 : \mathscr{X} \to \mathscr{M}^{n\times p}(\mathscr{X})$, $h_1 : \mathscr{X} \to \mathbb{R}^s$, and $k_{12} : \mathscr{X} \to \mathscr{M}^{s\times p}(\mathscr{X})$ are assumed to be real C^∞-functions of x. The H_∞ control problem, is described as finding a controller $K(x)$ which produces a control input such that in the closed-loop configuration satisfies,

$$\int_0^\infty ||z(t)||^2 dt \leq \gamma^2 \left[||x_0||^2 + \int_0^\infty ||w(t)||^2 dt \right], \qquad \forall w \in \mathscr{L}_2 \tag{6}$$

then we can say that the closed loop system has an $\mathscr{L}_2$-gain$\leq \gamma$. Furthermore, the closed-loop system should be stable.

A state-space system Σ is said to be dissipative with respect to the supply rate s if there exists a function $S : X \to \mathbb{R}^+$, called the storage function, such that for all $x_0 \in \mathscr{X}$, all $t_1 \geq t_0$, and all disturbances $w \in \mathscr{L}_2$.

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} s(w(t), z(t)) dt \tag{7}$$

The latter inequality is called the dissipation inequality. It expresses the fact that the ”stored energy” $S(x(t_1))$ of Σ at any future time t_1 is, at most, equal to the sum of the stored energy $S(x(t_0))$ at the present time t_0 and the total externally supplied energy is, $\int_{t_0}^{t_1} s(w(t), z(t)) dt$, during the time interval $[t_0, t_1]$. Hence, there can be no internal ”creation of energy”, only internal dissipation of energy is possible.

By choosing a supply rate:

$$s(w,z) = \frac{1}{2}\gamma^2 \|w\|^2 - \frac{1}{2}\|z\|^2, \quad \gamma \geq 0 \tag{8}$$

Σ is dissipative with respect to this supply rate if and only if there exists $S \geq 0$ such that for all $t_1 \geq t_0$, $x(t_0)$ and u valid the following:

$$\frac{1}{2}\int_{t_0}^{t_1} \left(\gamma^2 \|w\|^2 - \|z\|^2 \right) dt \geq S(x(t_1)) - S(x(t_0)) \tag{9}$$

It follows that the system Σ has $\mathscr{L}_2$-gain $\geq \gamma$ if it is dissipative with respect to the supply rate $s(w,z) = \frac{1}{2}\left(\gamma^2 \|w\|^2 - \|z\|^2\right)$.

We will consider a storage functions S as C^1 functions. By letting $t_1 \to t_0$ we see that (7) is equivalent to:

$$S_x \dot{x} \leq s(w, z(x,u)), \quad \forall x, u \tag{10}$$

with $S_x(x)$ denoting the vector of the partial derivatives $S_x(x) = \left(\frac{\partial S}{\partial x_1}(x), \ldots, \frac{\partial S}{\partial x_n}(x)\right)$. Furthermore, one can establish a direct link between dissipativity and Lyapunov stability. Assume now that $x^* \in \mathscr{X}$ is a strict local minimum of S. Then x^*, is a stable equilibrium of the unforced system $\dot{x} = f(x)$, i.e. $w = 0$, $u = 0$, with Lyapunov function $V(x) = S(x) - S(x^*) \geq 0$, for x around x^* [24]. According to (10) we can write for the above system as the following dissipation inequality:

$$V_x\left(f(x) + g_1(x)w + g_2(x)u\right) - \frac{1}{2}\gamma^2 \|w\|^2 + \frac{1}{2}\|z(x,u)\|^2 \leq 0 \tag{11}$$

maximizing with respect to w results in $w^* = \frac{1}{\gamma^2} g_1^T V_x^T$ while minimizing with respect to u results in $u^* = -g_2^T V_x^T$. Substituting these into the above inequality and assuming that $h_1(x)^T k_{12}(x) = 0$, yields the Hamilton Jacobi inequality (HJI):

$$\begin{aligned} HJI: \; V_x f(x) + \frac{V_x}{2}\left(\frac{1}{\gamma^2} g_1(x) g_1(x)^T - g_2(x) k_{12}(x)^T k_{12}(x) g_2(x)^T\right) V_x^T \\ + \frac{1}{2} h_1(x)^T h_1(x) \leq 0 \end{aligned} \tag{12}$$

which needs to be satisfied for all $x \in \mathscr{X}$. Thus, if exists a $V \geq 0$ which satisfies the latter inequality, then it is said that Σ has an $\mathscr{L}_2$-gain $\leq \gamma$. Therefore, sufficient condition for a system to have $\mathscr{L}_2$-gain is the existence of a controller $u(x) = K(x)$ which renders a dissipative closed loop system. By taking $t_0 = 0$ and assuming that $V(x(0)) \leq \gamma^2 \|x(0)\|^2$ then the dissiptivity implies that $\mathscr{L}_2$-gain $\leq \gamma$.

2.1 Sum of Square based Nonlinear H_∞ State-Feedback

Consider the following input-affine nonlinear time invariant system which is in a state dependent linear-like representation:

$$\begin{aligned} \dot{x} &= A(x)x^{\{d\}} + B_1(x)w + B_2(x)u \\ z &= C_1(x)x^{\{d\}} + D_{12}(x)u \\ y &= x \end{aligned} \tag{13}$$

where $x^{\{d\}}$ is an $N \times 1$ vector of monomials in x satisfying the following

Assumption 1 $x^{\{d\}} = 0$ *iff* $x = 0$

Remark 1. It should be noted that, given $f(x), h_1(x) \in \mathscr{P}^n$, the representation $f(x) = A(x)x^{\{d\}}$ and $h_1(x) = C_1(x)x^{\{d\}}$ is highly non-unique. Notice that for any $E(x)$ with $E(x)x^{\{d\}} = 0$, $A(x) + E(x)$ can also be used as a representation for $f(x)$. A special case of the representation corresponds to $x^{\{d\}} = x$, while $x^{\{d\}}$ can be selected to contain all the monomials in $f(x)$, i.e. $A(x)$ becomes a constant matrix.

Let $M(x)$ be a $N \times n$ polynomial matrix whose $(i,j)^{th}$ entry are given by

$$M_{ij} = \frac{\partial x_i^{\{d\}}}{\partial x_j}, \;\; i = 1,\ldots,N, \; j = 1,\ldots,n \tag{14}$$

Assumption 2 $C_1^T(x)D_{12}(x) = 0$ *and* $R_2(x) = D_{12}^T(x)D_{12}(x) > 0$

Theorem 1. *Consider system (13), if exists $X = X^T > 0$ and $Y(x)$ such that the following linear matrix inequality is satisfied while minimizing γ*

$$\begin{bmatrix} Y^T(x)B_2^T(x)M^T + M(x)B_2(x)Y(x) \\ \quad +XA^T(x)M^T + M(x)A(x)X & M(x)B_1(x) & Y^T(x) & XC_1^T(x) \\ * & -\gamma^2 I & 0 & 0 \\ * & * & -R_2(x) & 0 \\ * & * & * & -I \end{bmatrix} \leq 0, \tag{15}$$

then the control law $u = K(x)x^{\{d\}}$ stabilizes the system and achieves the H_∞ performance $\|z(x)\|_2 \leq \gamma \|w(x)\|_2$ with

$$K(x) = Y(x)X^{-1} \tag{16}$$

where $*$ indicates symmetric entries in a symmetric matrix.

Proof. Considering the closed-loop system of (13), a storage function $V(x) = x^{\{d\}T}(x)Px^{\{d\}}$ and controller matrix (16), then, according to the dissipation inequality (7) we obtain,

$$\begin{aligned} &A^T(x)M^TP + P(x)M(x)A(x) + PY^T(x)B_2^T(x)M^TP + P(x)M(x)B_2(x)Y(x)P \\ &+ \frac{1}{\gamma^2}PM(x)B_1(x)B_1^T(x)M^TP + C_1^T(x)C_1(x) + PY^T(x)D_{12}^T(x)D_{12}(x)Y(x)P \leq 0 \end{aligned} \tag{17}$$

multiplying both sides by $X = P^{-1}$, and applying the schur complement, then with the zero initial condition, the system is stable and the H_∞ performance is achieved as $\|z(x)\|_2 \leq \gamma\|w(x)\|_2$ with (16). □

3 Nonlinear H_∞ Output-Feedback

For the output-feedback suboptimal H_∞ control problem one wants to construct, if possible, for a given attenuation level $\hat{\gamma} \geq 0$ an output-feedback controller. We begin by synthesizing a dynamic observer-based controller using the output measurements. As before we consider an affine causal state space system defined on a smooth n-dimensional manifold $\mathscr{X} \subseteq \mathbb{R}^n$ in local coordinates $x = (x_1, \ldots, x_n)$:

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w_1 + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \\ y &= h_2(x) + k_{21}(x)w_2 \end{aligned} \tag{18}$$

the output $y \in \mathscr{Y} \subset \mathbb{R}^m$ is the measured output of the system, $h_2 : \mathscr{X} \to \mathbb{R}^m$ and $k_{21} : \mathscr{X} \to \mathscr{M}^{m\times r}(\mathscr{X})$ are assumed to be real C^∞-functions of x. The estimator and control law are modeled as

$$\begin{aligned} \dot{\xi} &= f(\xi) + g_1(\xi)w_1 + g_2(\xi)u + G(\xi)[y - h_2(\xi) - k_{21}(\xi)w_2] \\ u &= \alpha_2(\xi), \;\; \alpha_2(0) = 0 \end{aligned} \tag{19}$$

Substituting into the observer the optimal control law $u^* = \alpha_2(\xi)$, obtained from the state-feedback problem and the worst disturbance $w_2^* = \alpha_1(\xi)$, obtained as well from the state-feedback problem . Results in the following matrix formed dynamical equations

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix}}_{\dot{X}} = \underbrace{\begin{bmatrix} f(x) + g_1(x)\alpha_1(x) + \cdots \\ G(\xi)\,[h_2(x) + k_{21}(x)\alpha_1(x) - h_2(\xi) - k_{21}(\xi)\alpha_1(\xi)] + \cdots \end{bmatrix}}_{F(x,\xi)} \\ \begin{matrix} +g_2(x)\alpha_2(\xi) \\ +f(\xi) + g_1(\xi)\alpha_1(\xi) + g_2(\xi)\alpha_2(\xi) \end{matrix} \Bigg] + \underbrace{\begin{bmatrix} g_1(x) \\ G(\xi)k_{21}(x) \end{bmatrix}}_{g(x,\xi)} (w - \alpha_1(x)) \end{aligned} \tag{20}$$

Similar to the case of the state-feedback, dissipativity results in,

$$V_x\dot{x} + \|z\|^2 - \gamma^2\|w\|^2 = HJI + \|u - \alpha_2(x)\|^2_{R_2(x)} - \gamma^2\|w - \alpha_1(x)\|^2 \tag{21}$$

where the latter inequality can be written as,

$$V_x\,(f(x) + g_1(x)w + g_2(x)\alpha_2(\xi)) + \|z\|^2 - \gamma^2\|w\|^2 \leq \|v\|^2_{R_2(x)} - \gamma^2\|r\|^2 \tag{22}$$

where, $v = u - \alpha_2(x)$, $R_2(x) = k_{12}^T(x)k_{12}(x)$ and $r(x) = w - \alpha_1(x)$. Implementing the above supply rate such that the $\mathscr{L}_2$-gain will be sustained for the nonnegative C^1 storage function $W(X)$ yields,

$$W_X[F(X) + g(X)r] \leq \hat{\gamma}^2\|r\|^2 - \|v\|^2_{R_2(x)}\,. \tag{23}$$

While, substituting the essential supremum of $r(x)$ into the (23) results in the Hamilton Jacobi inequality for the unified system,

$$W_X F(X) + \frac{1}{4\gamma^2} W_X g(X) g^T(X) W_X^T + v^T(X) R_2(x) v(X) \leq 0. \quad (24)$$

This approach has essentially two disadvantages. The Hamilton Jacobi has twice as many independent variables as that of the state-feedback Hamilton Jacobi. The second disadvantage is the fact that the inequality is not convex since $G(\xi)$ is a design parameter. An alternative set of sufficient conditions for the solution of the problem are proposed in order to solve the problem of disturbance attenuation via measurement-feedback. The solution is based on an additional Hamilton Jacobi inequality which has the same number of independent variables as the Hamilton Jacobi inequality for the state-feedback problem. Assuming $W(X) = Q(x - \xi)$ we have,

$$\begin{aligned} HJI_q: \quad & Q_x[\hat{f}(x) - G(x)\hat{h}(x)] + \alpha_2^T(\xi) R_2(x) \alpha_2(\xi) \\ & + \frac{1}{4\hat{\gamma}^2} Q_x [g_1(x) - G(x) k_{21}(x)][g_1(x) - G(x) k_{21}(x)]^T Q_x^T \leq 0 \end{aligned} \quad (25)$$

where, $\hat{f}(x) = f(x) + g_1(x)\alpha_1(x)$, $\hat{h}(x) = h_2(x) + k_{21}(x)\alpha_1(x)$.

By completion to square of the HJI_q we obtain,

$$Q_x f_0(x) + \frac{1}{4\hat{\gamma}^2} Q_x g_0(x) g_0^T(x) Q_x^T + T_0 \leq 0 \quad (26)$$

where,

$$\begin{aligned} f_0(x) &= \hat{f}(x) - g_1(x) k_{21}^T R_1^{-1}(x) \hat{h}(x) \\ T_0(x) &= \alpha_2^T(x) R_2(x) \alpha_2(x) - \gamma^2 \hat{h}^T(x) R_1^{-1}(x) \hat{h}(x) \\ g_0(x) &= g_1(x)[I - k_{21}^T R_1^{-1}(x) k_{21}(x)] \\ R_1(x) &= k_{21}^T(x) k_{21}(x) \end{aligned} \quad (27)$$

This is valid if and only if

$$Q_x G(x) = [2\hat{\gamma}^2 \hat{h}^T(x) + Q_x g_1(x) k_{21}^T(x)] R_1^{-1}(x) \quad (28)$$

so that,

$$G(x) = \left(2\hat{\gamma}^2 L(x) + g_1(x) k_{21}^T(x)\right) R_1^{-1}(x) \quad (29)$$

if and only if Q satisfies $\hat{h}^T(x) = Q_x L(x)$, for some matrix $L(x)$ of smooth function of x.

3.1 Sum of Square based Nonlinear H_∞ Output-Feedback

Consider the following input-affine nonlinear time invariant system which is in the state dependent linear-like representation:

$$\begin{aligned} \dot{x} &= A(x)x^{\{d\}} + B_1(x)w_1 + B_2(x)u \\ z &= C_1(x)x^{\{d\}} + D_{12}(x)u \\ y &= C_2(x)x^{\{d\}} + D_{21}(x)w_2 \end{aligned} \tag{30}$$

where the dynamics of the estimator describes as,

$$\dot{\xi} = A(\xi)\xi^{\{d\}} + B_1(\xi)w_1 + B_2(\xi)u + G(\xi)[y - C_2(\xi)\xi^{\{d\}} - D_{21}(\xi)w_2] \tag{31}$$

Assumption 3 *The system matrices are such that $R_1(x) = D_{21}^T(x)D_{21}(x) > 0$ and $D_{21} : \mathscr{X} \to \mathscr{M}^{m\times m}(\mathscr{X})$, $\mathscr{W} \subset \mathscr{L}_2([t_0,\infty),\mathbb{R}^m)$ or $D_{21} : \mathscr{X} \to \mathscr{M}^m(\mathscr{X})$, $\mathscr{W} \subset \mathscr{L}_2([t_0,\infty),\mathbb{R})$.*

Theorem 2. *Consider system (30), if exists $T = T^T > 0$, such that the following linear matrix inequality is satisfied while minimizing $\hat{\gamma}$*

$$\begin{bmatrix} A_0^T(x)M^T(x)T + TM(x)A_0(x) \\ \quad -\gamma^2 C_2^T(x)R_1^{-1}(x)C_2(x) & PMB_2(x) & PMB_1(x)D_{21}^T(x) & TM(x)\hat{B}_1(x) \\ * & -R_2(x) & 0 & 0 \\ * & * & -\gamma^2 R_1(x) & 0 \\ * & * & * & -\hat{\gamma}^2 I \end{bmatrix} \leq 0 \tag{32}$$

then the measurement-feedback nonlinear H_∞ control problem for the system is solvable with the controller (16), (31) iff G(.) is selected as

$$G(x) = \left(2\hat{\gamma}^2 L(x) + B_1(x)D_{21}^T(x)\right) R_1^{-1}(x) \tag{33}$$

for some $n \times m$ smooth C^2 matrix function $L(x)$ which satisfies the condition

$$(M^T(x)T^{-1}x^{\{d\}} + x^{\{d\}T}T^{-1}M(x))L(x) = \hat{C}^T(x) \tag{34}$$

Where P,γ are obtained from the solution of the state-feedback problem (13), and $A_0(x)$, $\hat{B}_1(x)$,$\hat{C}^T(x)$ are defined as,

$$\begin{aligned} A_0(x) &= A(x) + \frac{1}{\gamma^2}B_1(x)B_1^T(x)M^T P - B_1(x)D_{21}^T(x)R_1^{-1}(x)\Big(C_2(x) + \frac{1}{\gamma^2}D_{21}(x)B_1^T(x)M^T P\Big) \\ \hat{B}_1(x) &= B_1(x)[I - D_{21}^T(x)R_1^{-1}(x)D_{21}(x)] \\ \hat{C}(x) &= \big(C_2(x) + \frac{1}{\gamma^2}D_{21}(x)B_1^T(x)P\big)x^{\{d\}} \end{aligned} \tag{35}$$

Proof. Suppose exists a negative definite function $\mathfrak{S}(x)$ for each nonzero x,

$$\begin{aligned}\mathfrak{S}(x) =& Q_x[A(x)x^{\{d\}} + B_1(x)\alpha_1(x) - G(x)\Big(C_2(x)x^{\{d\}} + D_{21}(x)w\Big)] + \alpha_2^T(x)R_2(x)\alpha_2(x)\\ &+ \frac{1}{4\hat{\gamma}^2}Q_x[B_1(x) - G(x)D_{21}(x)][B_1(x) - G(x)D_{21}(x)]^T Q_x^T \leq 0\end{aligned} \tag{36}$$

such that its Hessian matrix $\frac{\partial^2 \mathfrak{S}(x)}{\partial x^2}$ is nonsingular, where $Q(x)$ is a C^3 positive-definite function $Q : N_1 \subset \mathscr{X} \to \mathbb{R}_+$ locally defined in a neighborhood N_1 of $x = 0$, and vanishing at $x = 0$. In order for Q to satisfy *HJIq* (25), i.e. $Q(x-\xi) = W(X)$, is to proof that a function $\mathfrak{K}(x,\xi)$ is non-positive, for

$$\begin{aligned}\mathfrak{K}(x,\xi) =& W_X F(X) + v^T(X)R_2(x)v(X) + \frac{1}{4\hat{\gamma}^2}W_X g(X)g^T(X)W_X^T\\ =& \begin{bmatrix}W_x(X) & W_\xi(X)\end{bmatrix} F(X) + h^{eT}(X)R_2(x)h^e(X)\\ &+ \frac{1}{4\hat{\gamma}^2}\begin{bmatrix}W_x(X) & W_\xi(X)\end{bmatrix}\begin{bmatrix}g_1(x)g_1^T(x) & 0\\ 0 & G(\xi)R_1(x)G^T(\xi)\end{bmatrix}\begin{bmatrix}W_x^T(X)\\ W_\xi^T(X)\end{bmatrix}.\end{aligned} \tag{37}$$

By setting $e = x - \xi$ and defining

$$\mathfrak{F}(e,\xi) = \mathfrak{K}(x,\xi)\Big|_{x=e+\xi} \tag{38}$$

then by a second order Taylor expansion we obtain,

$$\mathfrak{F}(e,\xi) \approx \mathfrak{F}(0,\xi) + e^T\frac{\partial \mathfrak{F}(e,\xi)}{\partial e}\Big|_{e=0} + e^T\frac{\partial^2 \mathfrak{F}(e,\xi)}{\partial e^2}\Big|_{e=0} e \tag{39}$$

It can be shown that,

$$\mathfrak{F}(0,\xi) = \frac{\partial \mathfrak{F}(e,\xi)}{\partial e}\Big|_{e=0} = 0 \tag{40}$$

and that

$$\frac{\partial^2 \mathfrak{F}(e,\xi)}{\partial e^2}\Big|_{e,\xi=0} = \frac{\partial^2 \mathfrak{S}(x)}{\partial x^2}\Big|_{x=0}. \tag{41}$$

Since we set $\mathfrak{S}(x)$ to be non-positive we obtain that

$$\frac{\partial^2 \mathfrak{S}(x)}{\partial x^2}\Big|_{x=0} < 0 \tag{42}$$

which results in $\mathfrak{F}(e,\xi)$ being non-positive in the neighbourhood of $(e,\xi) = (0,0)$. Thus the function $Q(x-\xi)$ satisfies *HJIq* (25). By completion of the squares it can be shown that the function $\mathfrak{S}(x)$ satisfies the following inequality,

$$\begin{aligned}\mathfrak{S}(x) \geq & Q_x\left[A(x)x^{\{d\}} + B_1(x)\alpha_1(x) - B_1(x)D_{21}^T(x)R_1^{-1}(x)\left(C_2(x)x^{\{d\}} + D_{21}(x)w\right)\right] \\ & + \frac{1}{4\hat{\gamma}^2}Q_xB_1(x)\left[I - D_{21}^T(x)R_1^{-1}(x)D_{21}(x)\right]B_1^T(x)Q_x^T + \alpha_2^T(x)R_2(x)\alpha_2(x) \\ & - \hat{\gamma}^2\left(C_2(x)x^{\{d\}} + D_{21}(x)w\right)^T R_1^{-1}(x)\left(C_2(x)x^{\{d\}} + D_{21}(x)w\right)\end{aligned} \tag{43}$$

The latter inequality becomes an equality when,

$$Q_xG(x) = \left[2\hat{\gamma}^2\left(C_2(x)x^{\{d\}} + D_{21}(x)w\right)^T + Q_xB_1(x)D_{21}^T(x)\right]R_1^{-1}(x) \tag{44}$$

As a result we can conclude that in order for $\mathfrak{S}(x)$ to be non-positive, it is suffices to assume that the right hand side of inequality (43), which does not contain $G(x)$, is negative for each nonzero x. The right hand side of (43) can be written as,

$$Q_xA_0(x) + \frac{1}{4\hat{\gamma}^2}Q_xB_0(x)B_0^T(x)Q_x^T + \hat{T}_0(x) \leq 0 \tag{45}$$

where $A_0(x)$, $B_0(x)$ and $\hat{T}_0(x)$ are similarly defined in (27) . Assuming that $Q = x^{\{d\}T}(x)T^{-1}x^{\{d\}}$, and by the use of the schur complement we obtain (32) $\square$

Remark 2. It seems that the latter result is true for $G(x)$ and not for $G(\xi)$, although it can be easily shown that $G(x)$ and $G(\xi)$ are dual. This is done proving that

$$Q(e) = Q(x-\xi) = Q(\xi - x) = Q(-e) \tag{46}$$

i.e. (45), (33) can be written for ξ and not x. Thus, to show that $W(X) = Q(\xi - x)$ satisfies the Hamilton Jacobi inequality (24), is to show that the function $\mathfrak{K}(\xi,x)$ is non-positive. Therefore, similar to the proof which was presented before, by setting $e = -e$ and defining

$$\mathfrak{F}(-e,x) = \mathfrak{K}(\xi,x)\Big|_{\xi=-e+x} \tag{47}$$

results in $\mathfrak{F}(-e,x)$ being non-positive in the neighbourhood of $(-e,x) = (0,0)$. Thus the function $Q(\xi - x)$ satisfies the HJI (24).

If we conclude, in order to solve the H_∞ control via output-feedback with the use of SOS, the following convex optimization problems needs to be solved, for the state-feedback

$$\begin{aligned} &\text{minimize} && \gamma \quad \forall\, \zeta \\ &\text{subject to} && V(x) \quad \in \text{SOS} \\ & && -\zeta^T HJI\zeta \quad \in \text{SOS}\end{aligned} \tag{48}$$

and for the output measurement-feedback

$$\begin{array}{lll} \text{minimize} & \hat{\gamma} & \forall\, \zeta \\ \text{subject to} & Q(x) & \in \text{SOS} \\ & -\zeta^T HJI_q \zeta & \in \text{SOS} \end{array} \tag{49}$$

Thus, in order to implement the algorithm, ones needs to compute:

- The state-feedback problem (48), which result in $K(x)$, P and γ.
- The output-feedback problem (49) for $K(x)$, P and γ, which result in T and $\hat{\gamma}$.
- Compute a suitable matrix $L(x)$ which satisfies (34).
- Compute the estimators dynamic gain $G(x)$ from (33).
- Solve the estimator dynamics (31) for $w_1 = w_2 = \frac{1}{\gamma^2} B_1^T(\xi) P \xi^{\{d\}}$.

The following example will present the advantages of the use of SOS over the traditional solution; where by the use of SOS, the acceptable domain of suitable Lyapunov functions is much larger.

Example 1. Considering the following non linear system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -0.01 - 0.1x_1^2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.8 \end{bmatrix} w_1 + \begin{bmatrix} 0 \\ 1 + 0.13x_1^2 \end{bmatrix} u \\ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} 0.6 & 0.3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1.61 & 0 \\ 0 & 1.38 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{aligned} \tag{50}$$

Solving the output-feedback H_∞ problem for a *second* order Lyapunov function yields $\hat{\gamma} = 1.55$ and storage function,

$$Q(x) = 1.53x_1^2 + 1.3x_1x_2 + 1.62x_2^2$$

while the solution of output-feedback H_∞ problem for a *fourth* order Lyapunov function yields $\hat{\gamma} = 1.02$ and a storage function,

$$Q(x) = 0.38x_1^4 + 1.87x_1^2 + 1.93x_1x_2 + 1.13x_2^2$$

The above example reveals the advantages of the use of SOS, where a better disturbance attenuation closed-loop system is achieved.

4 Spacecraft Attitude Control

Consider a rigid body spacecraft which rotates around its center of mass under the influence of control and perturbations torques. Let $\mathscr{B}$ denote a spacecraft body frame, i.e., a Cartesian coordinates frame with the origin at the center of mass. Let $\mathscr{R}$ denote the Earth Centered Earth Inertial reference frame (ECEI). Let $\mathbf{q}$ denote the quaternion of rotation from $\mathscr{R}$ to $\mathscr{B}$, with vector part $\mathbf{e}$ and scalar part q [33, p.

758], and ω denote the angular velocity of $\mathscr{B}$ with respect to $\mathscr{R}$ expressed in $\mathscr{B}$. The rotational dynamics and kinematics of the rigid body spacecraft are governed by the following differential equations [33, Chap. 16]

$$\frac{d}{dt}\begin{bmatrix}\omega\\ \mathbf{e}\\ q\end{bmatrix} = \begin{bmatrix}-J^{-1}[\omega\times]J\omega\\ \frac{1}{2}\left(qI_3+[\mathbf{e}\times]\right)\omega\\ -\frac{1}{2}\mathbf{e}^T\omega\end{bmatrix} + \begin{bmatrix}J^{-1}\\ \mathbf{0}_{4\times 3}\end{bmatrix}\mathbf{T}_b \tag{51}$$

where J denotes the spacecraft tensor of inertia matrix in $\mathscr{B}$, $[\omega\times]$ denotes the cross-product matrix related to ω, and $\mathbf{T}_b$ is the vector of total external torques applied to the spacecraft, i.e.

$$\mathbf{T}_b = \mathbf{u}_b + \mathbf{w}_b \tag{52}$$

where $\mathbf{u}_b$ denote the 3×1 vector of control torques and $\mathbf{w}_b$ denote the 3×1 vector of disturbance torques. It is assumed that the Attitude Control System is equipped with a triad of three orthogonal reaction wheels, providing full control authority in all axes. The perturbation torques, modeled via $\mathbf{w}_b$, typically include the gravity gradient torque, the aerodynamic torque, a residual magnetic torque, and the solar pressure torque. Equation (51) is re-written as follows

$$\dot{\mathbf{x}} = f(\mathbf{x}) + G\mathbf{u}_b + G\mathbf{w}_b \tag{53}$$

where $\mathbf{x} \triangleq \{\omega, \mathbf{q}\}$. Notice that $f(\mathbf{x})$ is a polynomial function of the state variables. Also notice that f, G are C^k with $k \geq 2$, and that the unforced system has two equilibrium points:

$$(\omega, \mathbf{e}, q) = (\mathbf{0}, \mathbf{0}, \pm 1) \tag{54}$$

where both $\mathbf{q}_{1,2} = (0,0,0,\pm 1)$ correspond to the null attitude. It is assumed that the Attitude Determination and Control System is equipped with a suite of sensors that guarantee full observability of the state, such that $\mathbf{q}$ and ω can be estimated. As a first step, before applying more realistic assumptions, it is further assumed that the estimation errors can be neglected, i.e., that there is full state information. The attitude control objectives consist in globally stabilizing the system state Eq. (53) with respect to the equilibrium point $(\mathbf{0}, \mathbf{0}, 1)$, while attenuating the influence of the exogenous inputs $\mathbf{w}_b$ on the system dynamics.

4.1 Spacecraft Attitude Control Simulation

Considering the rotational dynamics and kinematics of a rigid body spacecraft by the differential equations governed in (51). The disturbance torque, w_b, is simulated as the sum of a torque caused by a impact collision and the aerodynamic drag. The impact collision, which is caused by the impact of a 1 gr particle, is described as a impulse function of 1.5 Nm with a duration of 0.1 sec. The particle hits the spacecraft at a velocity of 10 km/s and at a distance of 1 cm from the center of mass. The aerodynamic drag disturbance torque will be modeled as a first order

Markov process, which has the worst case magnitude for approximately 5% of the orbital period,

$$\begin{aligned} &T_d = \alpha_d \mathscr{N} + (1-\alpha_d)T_d \\ &\text{If } \ \|T_d\| > 1.89\cdot 10^{-3}\text{Nm} \\ &\text{Then } \ T_d = T_d \frac{2.89\cdot 10^{-3}}{\|T_d\|} \end{aligned} \tag{55}$$

where $\alpha_d = 1/6000$ is a filter constant, which determines the speed of the random walk, where a smaller value means a smaller speed. The variable $\mathscr{N}$ is a Gaussian white noise with a standard deviation of $0.75\cdot 10^{-5}$ Nm. The initial value T_d is chosen as a random unit vector with a magnitude of $2.89\cdot 10^{-5}$ Nm.

The measurement noise w_n is described as a finite energy Gaussian white noise. Let $\sigma_\omega(t)$ denote the time-varying variance intensity of the gyro's angler velocity measurements, which are equal to $0.25\cdot 10^{-3}$, and $\sigma_{\mathbf{q}}(t)$ as a time-varying variance intensity of the line-of-sight quaternion's measurements noise, which is equal to $0.25\cdot 10^{-4}$. The inertia matrix which was chosen is similar to a typical micro-satellite system and is equal to

$$J = \begin{bmatrix} 0.06 & 1\cdot 10^{-3} & 6\cdot 10^{-4} \\ 1\cdot 10^{-3} & 0.05 & 5\cdot 10^{-4} \\ 6\cdot 10^{-4} & 5\cdot 10^{-4} & 0.015 \end{bmatrix} \text{kgm}^2. \tag{56}$$

In order to use a quadratic Lyapunov function such that $x^{\{d\}} = x$ and such that the equilibrium vector is $\begin{bmatrix}\mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & 0\end{bmatrix}^T$ and not $\begin{bmatrix}\mathbf{0}_{1\times 3} & \mathbf{0}_{1\times 3} & 1\end{bmatrix}^T$ we shall perform a change in variables, i.e $\tilde{q} \triangleq q-1$ which result in the following tracking error dynamic system,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix}\omega \\ \tilde{\mathbf{e}} \\ \tilde{q}\end{bmatrix} &= \begin{bmatrix} -J^{-1}[\omega\times]J & \mathbf{0}_{3\times 4} \\ 0.5\cdot\begin{bmatrix}\mathbf{I}_{3\times 3} \\ \mathbf{0}_{3\times 1}\end{bmatrix} & 0.5\cdot\begin{bmatrix}-[\omega\times] & \omega \\ -\omega^T & 0\end{bmatrix}\end{bmatrix}\begin{bmatrix}\omega \\ \tilde{\mathbf{e}} \\ \tilde{q}\end{bmatrix} + \begin{bmatrix}\mathbf{1}_{3\times 1} \\ \mathbf{0}_{4\times 1}\end{bmatrix} w_b + \begin{bmatrix} J^{-1} \\ \mathbf{0}_{4\times 3}\end{bmatrix}\mathbf{u}_b \\ z &= \begin{bmatrix}\mathbf{u}_b \\ \tilde{\mathbf{e}} \\ \tilde{q}\end{bmatrix} = C_1\begin{bmatrix}\omega \\ \tilde{\mathbf{e}} \\ \tilde{q}\end{bmatrix} + D_{12}\mathbf{u}_b \\ y &= \mathbf{I}_{7\times 7}\begin{bmatrix}\omega \\ \tilde{\mathbf{e}} \\ \tilde{q}\end{bmatrix} + \begin{bmatrix}\frac{\sigma_{\mathbf{q}}}{\sigma_\omega}{}_{3\times 1} \\ \sigma_{\mathbf{q}3\times 1}\end{bmatrix} w_n \end{aligned} \tag{57}$$

where $\mathbf{1}$ denotes a vector of ones. The measurements of $\begin{bmatrix}\tilde{\mathbf{e}} & \tilde{q}\end{bmatrix}^T$ are obtained from the line-of-sight quaternion's measurements $\begin{bmatrix}\mathbf{e} & q\end{bmatrix}^T$. Several simulation were performed, in order to measure the performance of the H_∞ output-feedback controller which was derived. The attenuation level which was obtained from the semi-definite optimization problem was $\gamma = 0.08$ and $\hat{\gamma} = 4.9$, where the matrix L was chosen by minimizing its Euclidean norm, while satisfying (34). In addition the Lyapunov functions which where obtained are,

$$\begin{aligned} V(\mathbf{x}) =& 114.2q^2 + 114.26e_1^2 + 114.23e_2^2 + 0.014qe_3 + 114.2e_3^2 + 4.07q\omega_1 \\ &+ 0.1e_3\omega_1 + 0.07\omega_1^2 + 0.11q\omega_2 + 0.0801e_3\omega_2 + 0.03\omega_1\omega_2 + 0.13\omega_2^2 \\ &+ e_2\left(0.14q + 0.0409e_3 + 0.11\omega_1 + 0.067\omega_2 + 0.06\omega_3\right) \\ &+ e_1\left(0.09q + 0.099e_2 + 0.11e_3 + 0.122\omega_1 + 0.15\omega_2 + 0.11\omega_3\right) \\ &+ 0.13q\omega_3 + 0.053e_3\omega_3 + 0.09\omega_1\omega_3 + 0.08\omega_2\omega_3 + 0.08\omega_3^2 \end{aligned} \tag{58}$$

$$\begin{aligned} Q(\mathbf{x}) =& 15.61q^2 + 14.57e_1^2 + 15.55e_2^2 + 0.05qe_3 + 14.56e_3^2 + 0.12q\omega_1 \\ &+ 0.15e_3\omega_1 + 0.01\omega_1^2 + 0.064q\omega_2 + 0.07e_3\omega_2 + 0.08\omega_1\omega_2 + 0.02\omega_2^2 \\ &+ e_2\left(0.11q + 0.14e_3 + 0.81\omega_1 + 0.05\omega_2 + 0.042\omega_3\right) \\ &+ e_1\left(0.19q + 0.14e_2 + 0.14e_3 + 0.09\omega_1 + 0.13\omega_2 + 0.12\omega_3\right) + 0.11q\omega_3 \\ &+ 0.92e_3\omega_3 + 0.11\omega_1\omega_3 + 0.06\omega_2\omega_3 + 0.07\omega_3^2 \end{aligned} \tag{59}$$

It is of great interest to compare the H_∞ performance with a standard proportional controller [34, 14], and with an optimal nonlinear control law, for example the state dependent Riccati equality (SDRE) controller [6]. The proportional controller, PD_∞, was derived based on the H_∞ controller. While, in both cases the SDRE and the proportional controller used the H_∞ estimator. The initial conditions for the simulations were considered as $\left[1\ 0.5\ -0.5\ 0\ 1\ 0\ 0\right]^T$. An extended Kalman filter (EKF) was implemented as well for the SDRE controller, but was not capable to cope with the disturbances and as a result did not converge. It can be seen from Fig. 1 and Fig. 2 that the H_∞ controller achieves a better disturbance attenuation closed-loop system than the SDRE and the proportional controllers. Moreover, the measurement noise is better attenuated, and the control effort is reduced, despite the fact that they are both based on the H_∞ estimator.

5 Conclusions

A novel computational scheme was presented in order to solve the output-feedback H_∞ control problem for a class of nonlinear systems with polynomial vector field. By converting the resulting Hamilton-Jacobi inequalities from rational forms to their equivalent polynomial forms, we overcome the non-convex nature and numerical difficulty. Using quadratic Lyapunov functions, both the state-feedback and output-feedback problems were reformulated as semi-definite optimization conditions, while locally tractable solutions were obtained through sum of squares (SOS) programming. A numerical example and a spacecraft attitude control simulation showed that the proposed computational scheme result in a better disturbance attenuation closed-loop system, and more robust, while compared to standard methods.

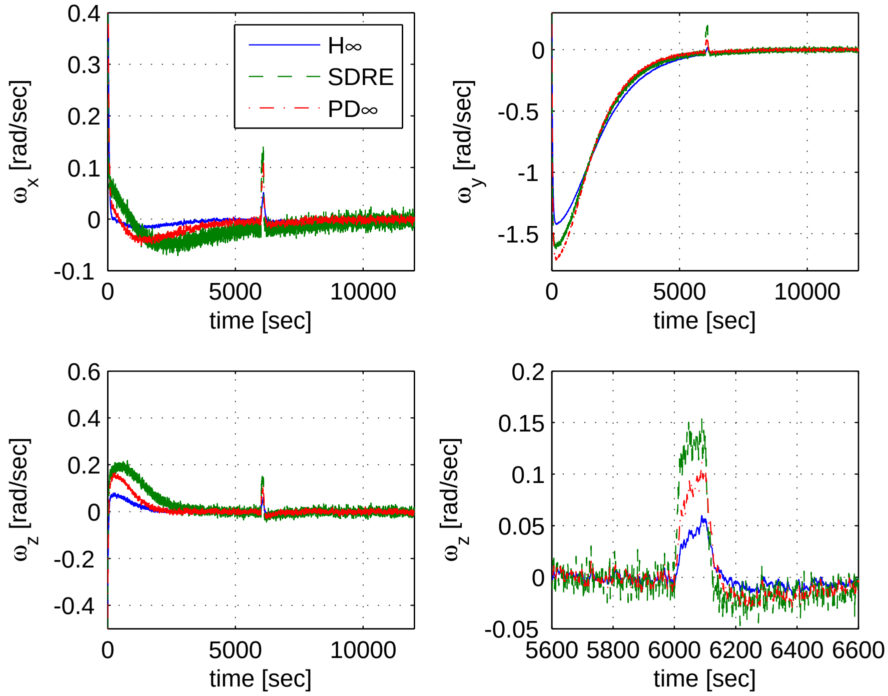

Fig. 1 Closed-loop angular velocities based on three different controllers. All three controllers use the nonlinear H_∞ estimator. It can be clearly seen that the H_∞ controller achieves a better disturbance attenuation closed-loop system.

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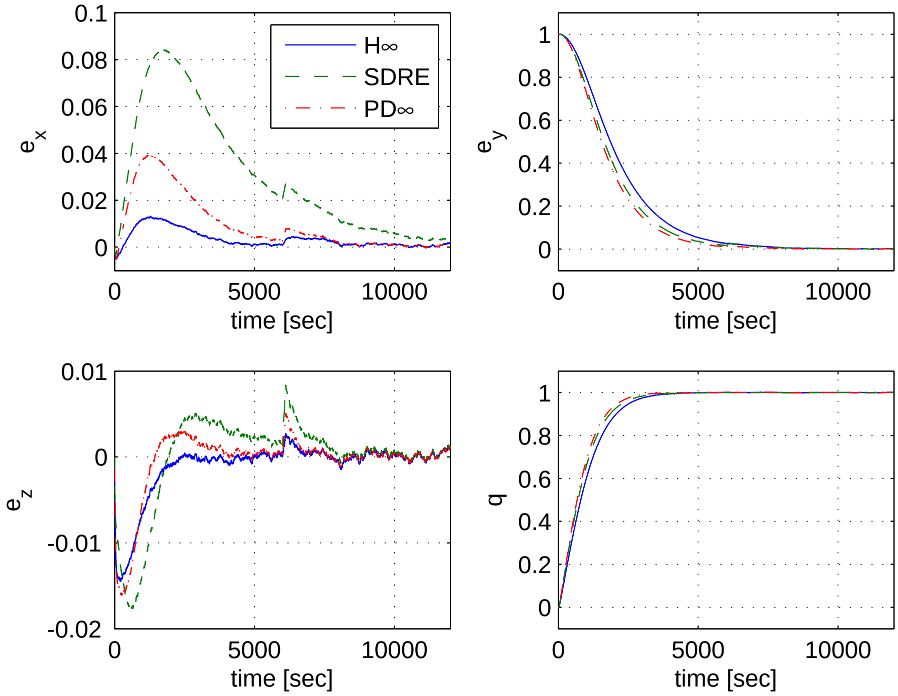

Fig. 2 Closed-loop attitude quaternion time histories for three different controllers. All three controllers use the nonlinear H_∞ estimator.

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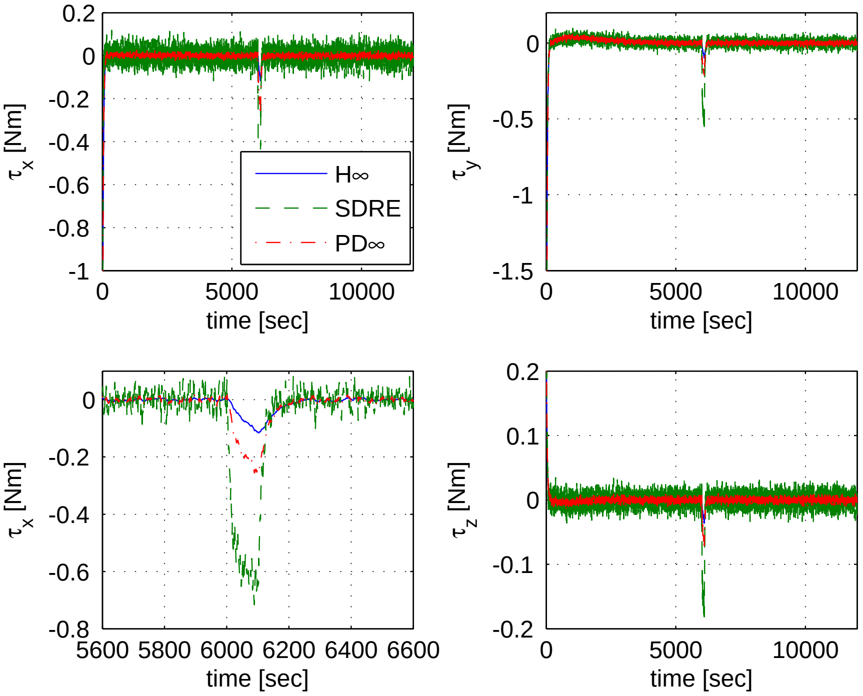

Fig. 3 Control signals time histories obtained from three different controllers.

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